

Well-Formed Separator Sequences, with an Application to Hypergraph Drawing

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Abstract

Given a hypergraph \mathcal{H} , the PLANAR SUPPORT problem asks whether there is a planar graph G on the same vertex set as \mathcal{H} such that each hyperedge induces a connected subgraph of G . PLANAR SUPPORT is motivated by applications in graph drawing and data visualization. We show that PLANAR SUPPORT is fixed-parameter tractable when parameterized by the number of hyperedges in the input hypergraph and the outerplanarity number of the sought planar graph. To this end, we develop novel structural results for r -outerplanar triangulated disks, showing that they admit sequences of separators with structural properties enabling data reduction. This allows us to obtain a problem kernel for PLANAR SUPPORT, thus showing its fixed-parameter tractability.

1 Introduction

A *support* for a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a graph G on the same vertex set \mathcal{V} such that, for each hyperedge $e \in \mathcal{E}$, the subgraph of G induced by the vertices in e is connected. If there is no restriction on the support, then any given hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ has a support, namely the clique on \mathcal{V} . For a graph property Π , the problem of deciding whether a given hypergraph \mathcal{H} has a support that satisfies Π —shortly, a Π -support—has been studied by various research communities for numerous properties Π . This problem has, among others, applications in graph drawing, databases, and social and overlay networks [1, 2, 4–7, 10, 17, 18, 22, 23]. The studied graph properties include: having minimum number of edges, being a path, a cycle, a tree, having bounded treewidth, being planar, and being r -outerplanar. For some of these properties, the problem is known to be solvable in polynomial time (*e.g.*, path [5, 19], cycle [5], tree [2, 17, 23]), for some it is known to be NP-hard (*e.g.*, minimum number of edges [9], planar [17], 2-outerplanar [5]), and for some its complexity remains unresolved (*e.g.*, outerplanar [5]).

Planar supports. Perhaps the majority of the work on hypergraph support problems is related to hypergraph drawing or representation. Here, one seeks a plane drawing of the hypergraph that captures the relations among its vertices—stipulated by its hyperedges, while revealing these relations elegantly via the drawing of the hypergraph. One method for drawing hypergraphs is to draw them as *vertex-based Venn diagrams* [17], also referred to as *subdivision drawings* [18]. A subdivision drawing of a hypergraph is a plane subdivision such that each vertex of the hypergraph corresponds uniquely to a face of the subdivision, and for each hyperedge the union of all the faces corresponding to the vertices in the hyperedge forms a connected region. A hypergraph has

a subdivision drawing if and only if it has a planar support [18]. Deciding whether a hypergraph has a planar support is NP-complete [17]. We study the parameterized complexity of the problem parameterized by the combination of the number of hyperedges in the hypergraph and the outerplanarity r of the sought support:

PLANAR SUPPORT

Input: A hypergraph \mathcal{H} with n vertices and m hyperedges, and an $r \in \mathbb{N}$.

Question: Does \mathcal{H} have a planar support of outerplanarity at most r ?

Parameter: The number m of hyperedges in \mathcal{H} and r combined.

Known results. PLANAR SUPPORT is NP-hard for $r = 2$ [5]. Buchin et al. [5] give an NP-hardness reduction for $r = 3$ that transforms a 3-SAT instance into a hypergraph \mathcal{H} . By inspecting the construction of \mathcal{H} , it can be easily verified that \mathcal{H} either has a 3-outerplanar support or no planar support at all. Thus, the reduction of Buchin et al. [5] implies that, for every fixed $r > 3$, PLANAR SUPPORT is NP-hard as well: in the reduction, simply add a fixed r -outerplanar graph to \mathcal{H} ; the resulting hypergraph has an r -outerplanar support if and only if \mathcal{H} has a 3-outerplanar support. The (classical) complexity of PLANAR SUPPORT for $r = 1$ remains open [5].

An underlying assumption for several results in the literature pertaining to PLANAR SUPPORT (e.g., Mäkinen [20, p. 179], Buchin et al. [5, p. 346], Kaufmann et al. [18, p. 399]) has been that the hypergraph is *twinless*, that is, does not contain two vertices (*twins*) such that the set of hyperedges containing the first is the same as that containing the second. (Twins were referred to as “equivalent” vertices by Buchin et al. [5].) The intuition behind this assumption is that a twin does not affect the instance because whatever can be “achieved” by a vertex can be achieved by its twin. In Section 6, we demonstrate that this assumption changes the landscape of PLANAR SUPPORT completely: we exhibit hypergraphs with twins that admit (r -outer)planar supports but depriving them of their twins results in hypergraphs with no (r -outer)planar supports. This illustrates the important role that twins play in realizing (r -outer)planar supports for hypergraphs. Indeed, the presence of twins makes PLANAR SUPPORT much more challenging: one can easily show PLANAR SUPPORT for twinless hypergraphs to be fixed-parameter tractable (FPT), whereas showing FPT in general hypergraphs is much more demanding.

It can be shown that, for each value of the parameter m , there is an (unknown and distinct) algorithm solving PLANAR SUPPORT in $f(m) \cdot \text{poly}(n)$ time for some function f . In other words, PLANAR SUPPORT is *non-uniformly* FPT when parameterized by m and hence, also when parameterized by m and r combined. To obtain this result, one can use the known machinery of *well-quasi orderings* [8] to prove that each yes-instance of PLANAR SUPPORT contains some minimal yes-instance, and that the number and size of minimal yes-instances depends only on m . We outline the proof in Section 7. Note, however, that even undecidable problems can be non-uniformly FPT. Therefore, non-uniform FPT results in general are unimplementable and extensive research has focused on making non-uniform FPT results uniform (e.g., numerous FPT results that can be obtained using graph minor theory).

Our contributions. We present an algorithm that decides a given instance of PLANAR SUPPORT in $f(m, r) \cdot \text{poly}(n)$ time, where f is *explicitly given*. This implies that PLANAR SUPPORT is strongly uniformly FPT, which is a strong improvement over the above-mentioned non-uniform FPT result, and a necessary step in order to get applicable algorithms.

We prove that PLANAR SUPPORT is FPT by providing a problem kernel. Notably, the number of hyperedges is perhaps the most natural parameter to study, as already observed in previous work [6, 16]. Also note that a problem kernel with respect to the studied parameter combination is a stronger result than having for each fixed r a problem kernel with respect to the parameter “number of hyperedges”. The main ingredient of the problem kernel is the non-trivial observation that, indeed, removing one of *sufficiently many* twins does not affect the instance. To obtain the problem kernel, based on the crucial observation that, without loss of generality, we can focus

on triangulated disks, we prove a general structural result about separators of r -outerplanar triangulated disks. This is of independent interest: Given an embedding of an r -outerplanar triangulated disk G ($r \geq 1$) on n vertices, we show that one can construct in polynomial time a sequence of separators for G , which we refer to as a *well-formed separator sequence* and whose length is some increasing, unbounded function in r and n .

We formally introduce well-formed separator sequences in Section 3 and compare them to other separator families found in the literature. Their structural properties make them amenable to a *gluing* operation, which is also introduced in Section 3. Gluing removes the subgraph of G “between” any two separators in the embedding, and identifies the separators. We show that gluing any two separators in a well-formed separator sequence preserves the r -outerplanarity of G . To apply this toolkit to PLANAR SUPPORT, we show that if the number of vertices in the hypergraph \mathcal{H} is “large” with respect to the parameter, then there are two separators in the planar support (if one exists) such that the subgraph between the two separators is “redundant” (*i.e.*, does not have any effect on the connectivity of the hyperedges), a property that we capture using the notion of *separator signatures* (Section 4). The above allows us to conclude that if an r -outerplanar support for \mathcal{H} exists, then a support whose size is upper-bounded by a function of the parameter must exist as well. This gives a problem kernel and, as a consequence, an FPT algorithm for PLANAR SUPPORT. Section 5 provides the technical construction of well-formed separator sequences.

2 Preliminaries

We use standard terminology from graph theory [24] and parameterized complexity [8, 13, 21].

Graphs. Unless stated otherwise, all graphs are without parallel edges or loops. A *cut-vertex* (resp. *cut-edge*) in a connected graph G is a vertex v (resp. an edge e) such that $G - v$ (resp. $G - e$) is disconnected. A connected graph G is *biconnected* if no vertex in G is a cut-vertex. The *blocks* of a graph G are its maximal biconnected subgraphs, its cut-edges, and its isolated vertices.

r -Outerplanar disks. A *plane graph* $G = (V, E)$ is a planar graph given with a fixed embedding in the plane. The *layer decomposition* of G with respect to the embedding is a partition of V into layers $L_1 \uplus \dots \uplus L_r$ is defined inductively as follows. Layer L_1 is the set of vertices that lie on the outer face of G , and layer L_i is the set of vertices that lie on the outer face of $G - \bigcup_{j=1}^{i-1} L_j$ for $1 < i \leq r$. The graph G is called *r -outerplanar* if it has an embedding with a layer decomposition consisting of at most r layers. If $r = 1$, then G is simply said to be *outerplanar*. A plane graph G is said to be *triangulated* if each face of G , including the outer face, is a triangle, and G is said to be a *triangulated disk* if its outer face is a simple cycle (not necessarily a triangle), and all its inner faces are triangles [3]. It is easy to see that the vertices on the outer face of a biconnected r -outerplanar graph form a simple cycle. In most sections of this paper, we will be working with a fixed r -outerplanar triangulated disk G , that is, we implicitly fix an embedding of G . When the context is clear, we will often abuse the notation and use L_1 to refer to the simple cycle that delimits the outer face of G . It is known that any vertex v in layer L_i , $i > 1$, of an r -outerplanar triangulated disk G has a neighbor in layer L_{i-1} [3].

Hypergraphs. A *hypergraph* $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ consists of a vertex set $\mathcal{V} = V(\mathcal{H})$ and an edge set $\mathcal{E} = E(\mathcal{H})$ such that $e \subseteq \mathcal{V}$ for every $e \in \mathcal{E}$. Throughout this work, we denote $n := |\mathcal{V}|$ and $m := |\mathcal{E}|$. The *size* of a hyperedge is the number of vertices in it. Unless stated otherwise, we assume that hypergraphs do not contain hyperedges of size at most 1 or multiple copies of the same hyperedge. For a vertex $v \in \mathcal{H}$, we denote $\mathcal{E}(v) := \{e \in \mathcal{H} \mid v \in e\}$. A vertex v *covers* a vertex u if $\mathcal{E}(u) \subseteq \mathcal{E}(v)$. Two vertices $u, v \in \mathcal{V}$ are *twins* if $\mathcal{E}(v) = \mathcal{E}(u)$. Clearly, the relation \mathcal{R} on \mathcal{V} defined by $\forall u, v \in \mathcal{V}, u\mathcal{R}v \iff \mathcal{E}(u) = \mathcal{E}(v)$ is an equivalence relation. We write $[u]_{\mathcal{R}}$ to

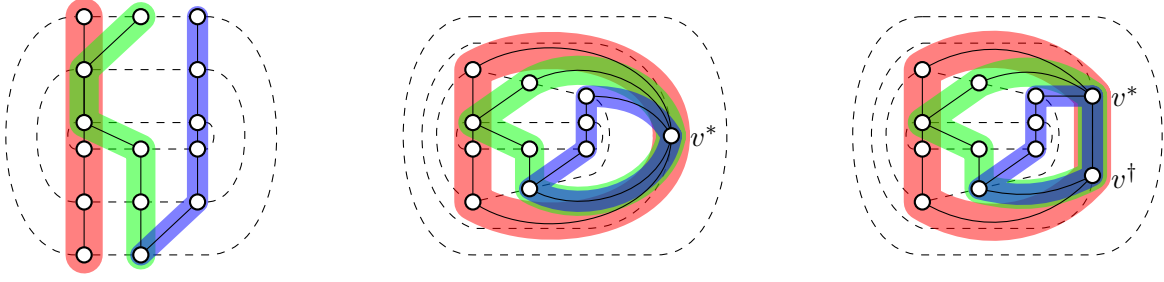


Figure 1: Three well-formed separator sequences according to Definition 3.1, one satisfying Property (va), one satisfying Property (vb) with $v^* = v^\dagger$, and one satisfying Property (vb) with $v^* \neq v^\dagger$. Dashed lines are the layers, solid lines are the edges, and shaded areas in each drawing are the separators, where different separators are distinguished using different shades.

denote the *twin class* of a vertex $u \in \mathcal{V}$ under the above relation \mathcal{R} . Removing a vertex set S from a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ results in the hypergraph $\mathcal{H} - S := (\mathcal{V} \setminus S, \mathcal{E}')$ where \mathcal{E}' is obtained from $\{e \setminus S \mid e \in \mathcal{E}\}$ by removing the empty set and singleton sets. We use $\mathcal{H}[S] := \mathcal{H} - (\mathcal{V} \setminus S)$ and $\mathcal{H} - v := \mathcal{H} - \{v\}$.

Parameterized complexity. A *parameterized problem* is a set of instances of the form (\mathcal{I}, k) , where $\mathcal{I} \in \Sigma^*$ for a finite alphabet Σ , and $k \in \mathbb{N}$ is the *parameter*. A parameterized problem Q is *fixed-parameter tractable*, shortly FPT, if there exists an algorithm that on input (\mathcal{I}, k) decides if (\mathcal{I}, k) is a yes-instance of Q in $f(k)|\mathcal{I}|^{O(1)}$ time, where f is a computable function independent of $|\mathcal{I}|$. A parameterized problem Q is *kernelizable* if there exists a polynomial-time self-reduction that maps an instance (\mathcal{I}, k) of Q to another instance (\mathcal{I}', k') of Q such that: (1) $|\mathcal{I}'| \leq \lambda(k)$ for some computable function λ , (2) $k' \leq \lambda(k)$, and (3) (\mathcal{I}, k) is a yes-instance of Q if and only if (\mathcal{I}', k') is a yes-instance of Q . The instance (\mathcal{I}', k') is called the *problem kernel* of (\mathcal{I}, k) . It is well known that a parameterized problem is fixed-parameter tractable if and only if the problem is kernelizable.

3 Well-formed separator sequences

In this section, we introduce well-formed separator sequences, state our main structural contribution, and compare it to results of similar nature in the literature. Moreover, we introduce the gluing operation that well-formed separator sequences are amenable to.

The separators in a well-formed separator sequence all have the same number of vertices and are either all induced paths or all induced cycles. Moreover, the separators stretch along consecutive layers of the r -outerplanar graph such that each separator contains at most two vertices from each layer and there is a one-to-one layer-correspondence between the vertices of the separators in the sequence (see Figure 1 for illustration).

Definition 3.1 (Well-formed separator sequence). Let $G = (V, E)$ be a graph with a fixed plane embedding with layers L_1, \dots, L_r . A *well-formed separator sequence* of length t and width p for G is a sequence $(A_1, S_1, B_1), \dots, (A_t, S_t, B_t)$ satisfying the following properties:

Linear Separation: For each $i \in \{1, \dots, t\}$,

- (i) $V = A_i \cup B_i$,
- (ii) there is no edge between $A_i \setminus B_i$ and $B_i \setminus A_i$,
- (iii) $S_i = A_i \cap B_i$, $|S_i| = p$, and
- (iv) $A_i \subsetneq A_{i+1}$ and $B_i \supsetneq B_{i+1}$.

Simple Shape:

- (v) One of the following two conditions holds:

- a) for all $i \in \{1, \dots, t\}$, vertex set S_i induces a path $(v_{i,1}, \dots, v_{i,p'})$ with $v_{i,1}, v_{i,p'} \in L_1$ (in this case, $p' = p$); or
 - b) $L_1 \subseteq A_1$ and, for all $i \in \{1, \dots, t\}$, vertex set S_i induces a cycle $(v^*, v_{i,1}, \dots, v_{i,p'}, v^\dagger)$, where v^* and v^\dagger are on the layer of minimum index that intersects S_i and, possibly, $v^* = v^\dagger$ (in this case, $p' \in \{p-1, p-2\}$).
- (vi) For $1 \leq i, j \leq t$ and $1 \leq k, \ell \leq p'$, if $v_{i,k} = v_{j,\ell}$, then $k = \ell$.

Layering:

- (vii) S_i contains at most two vertices from each layer of G , and
- (viii) for $1 \leq i, j \leq t$ and $1 \leq k \leq p'$, vertex $v_{i,k}$ and vertex $v_{j,k}$ are on the same layer.

Our main structural contribution in this paper is the following theorem, proved in Section 5.

Theorem 3.2. *Any r -outerplanar triangulated disk with n vertices contains a well-formed separator sequence of length at least $\lfloor 2^r \sqrt{\log(n)}/6^r \rfloor$ and width at most $2r$.*

There are several well-known approaches for constructing separators for planar/ r -outerplanar graphs satisfying some of the properties of well-formed separator sequences. For example, an r -outerplanar graph G has treewidth at most $3r - 1$ and branchwidth at most $2r$ [3], and thus, we can construct separator families that satisfy Properties (i) to (iii) and have width $3r$ or $2r$ from the respective tree or branch decompositions: each bag of a tree decomposition for G is a separator for G , and each edge in a branch decomposition corresponds to a separator. Moreover, since branch decompositions are trees of bounded degree, there is an arbitrarily long path in a branch decomposition of a sufficiently large graph, and thus, an arbitrarily long sequence of separators additionally satisfying Property (iv). However, arbitrarily large subsequences satisfying our key Properties (v) to (viii) may not be extracted from a tree/branch decomposition of G .

Layered separators [11, 12] yield, for r -layer embeddings of sufficiently large graphs, arbitrarily long separator sequences of bounded width that satisfy Properties (iv) and (vii). The ones of Dujmović [11] yield a sequence satisfying Properties (i), (iii), (iv) and (vii), but only a weaker variant of Property (ii), namely, that there is no edge between $(A_i \cap B_{i-1}) \setminus B_i$ and $B_i \setminus (A_i \cap B_{i-1})$. That is, each separator S_i is a separator for $G[B_{i-1}]$ but not necessarily for G . The (slightly different) ones of Dujmović, Morin, and Wood [12] yield a sequence satisfying Properties (i), (ii), (iv) and (vii), if one changes Property (iii) so that $S_i = (A_i \cap B_i) \setminus A_{i-1}$. That is, the separator $A_i \cap B_i$ might use more than two vertices of a layer if these vertices are in A_{i-1} . Neither variant of the layered separators in [11, 12] satisfies the key Properties (v) and (viii) of well-formed separator sequences.

Gluing separators of a well-formed separator sequence. In the following, we show a property of well-formed separator sequences exploited in our algorithm for PLANAR SUPPORT. Consider the following operation for a given well-formed separator sequence: Pick two arbitrary separators (A_i, S_i, B_i) and (A_j, S_j, B_j) in the sequence; remove everything in the graph that is contained “between” the separators, that is, keep only $A_i \cup B_j$; and glue the two separators S_i and S_j by identifying their vertices.

Definition 3.3 (Gluing). Let G be an r -outerplanar triangulated disk, and let $T_i = (A_i, S_i, B_i)$ and $T_j = (A_j, S_j, B_j)$, $i < j$, be two separators of a well-formed separator sequence of width p for G . We define $G(T_i \circ T_j)$ to be the graph obtained by taking the disjoint union of $G[A_i]$ and $G[B_j]$ and identifying each $v_{i,k}$ in S_i with $v_{j,k}$ from S_j , for $k = 1, \dots, p$.

As we show below, well-formed separator sequences behave nicely with respect to the gluing operation in the sense that the resulting graph is again r -outerplanar.

Lemma 3.4. $G(T_i \circ T_j)$ is r -outerplanar.

Proof. First, observe that if S_i is trivial in the sense that $A_i = S_i$, then the lemma holds trivially since the gluing operation degenerates to taking a subgraph of G . By symmetry, the same holds if $S_j = B_j$. If S_i and S_j are nontrivial, then we distinguish two cases based on whether the two separators induce paths or cycles.

Case 1: $G[S_i]$ and $G[S_j]$ are paths. Consider the fixed embedding of G . Since S_i is an induced path with its two endpoints in L_1 and no other vertices in L_1 , it separates the region enclosed by L_1 into two regions that intersect only in the vertices and edges of $G[S_i]$. Towards a contradiction, assume that one of these two regions contains vertices from both $A_i \setminus S_i$ and $B_i \setminus S_i$. Then, since $G[S_i]$ is an *induced* path, there is a face in this region that contains vertices from $A_i \setminus S_i$ and $B_i \setminus S_i$. Since this face is a triangle, there is an edge between $A_i \setminus S_i$ and $B_i \setminus S_i$; this contradicts Property (ii). Thus, one of the two regions contains the vertices of $A_i \setminus S_i$ and the other one contains the vertices of $B_i \setminus S_i$. Therefore, deleting all vertices in the region containing $B_i \setminus S_i$ gives an embedding of $G[A_i]$ in which all vertices of S_i and all vertices of $A_i \cap L_1$ lie on the boundary of the outer face. The same statement holds for $G[B_j]$, that is, there is an embedding of $G[B_j]$ such that all vertices of $S_j \cup (B_j \cap L_1)$ lie on the outer face of $G[B_j]$. Moreover, the same is true for the disjoint union G' of $G[A_i]$ and $G[B_j]$ (using translation, we can assume that the embedding of $G[B_j]$ is strictly to the right of $G[A_i]$). Now, for each ℓ , $1 \leq \ell \leq p$, add the edge $\{v_{i,\ell}, v_{j,\ell}\}$. The resulting graph is planar: the edge $\{v_{i,1}, v_{j,1}\}$ is between different connected components of G' ; thus it can be added without destroying planarity. The resulting outer face either has a counterclockwise face walk which contains the subsequence $(v_{j,p}, v_{j,p-1}, \dots, v_{j,2}, v_{j,1}, v_{i,1}, v_{i,2}, \dots, v_{i,p-1}, v_{i,p})$ or this situation can be achieved by suitable reflections of $G[A_i]$ or $G[B_j]$ along the horizontal axis. Now, adding $\{v_{i,2}, v_{j,2}\}$ replaces the old outer face by two new faces; one of these faces has a counterclockwise face walk with the subsequence $(v_{j,p}, v_{j,p-1}, \dots, v_{j,3}, v_{j,2}, v_{i,2}, v_{i,3}, \dots, v_{j,p-1}, v_{j,p})$. Hence, $(v_{i,3}, v_{j,3})$ can be added in the same way, again creating two new faces. This process can be repeated until finally the edge $\{v_{i,p}, v_{j,p}\}$ is added. The resulting graph is planar and by contracting each of the p edges added to G' we obtain again a planar graph. This graph is exactly $G(T_i \circ T_j)$: after these contractions, the neighborhood of each $v_{i,\ell}$ is exactly the union of $N(v_{i,\ell}) \cap (A_i \setminus S_i)$ and $N(v_{j,\ell}) \cap (B_j \setminus S_j)$ plus $v_{i,\ell-1}$ and $v_{i,\ell+1}$ if they exist. All neighborhoods in $A_i \setminus S_i$ remain the same in G and the constructed graph, and all neighborhoods in $B_j \setminus S_j$ remain the same except that $v_{j,\ell}$ is replaced by $v_{i,\ell}$ in each neighborhood.

It remains to show r -outerplanarity. First, observe that the vertices of $A_i \cap L_1$ and of $(B_j \setminus S_j) \cap L_1$ are on the boundary of the outer face of $G(T_i \circ T_j)$ (if it is embedded as described above). This also implies that, in $G(T_i \circ T_j)$, each vertex of S_i is in the same layer as in G . It remains to show that also each vertex v of A_i is in the same layer as in G . To this end, we exploit that $G(T_i \circ T_j)$ is a triangulated disk and, thus, that a vertex v is in L_i if and only if a shortest path from v to L_1 has length exactly i [3].

Take any path witnessing that $v \in A_i$ is in layer L_q of G . If this path contains no vertex from S_i , then this path is also present in $G[A_i]$. If this path contains some vertex $v_{i,k}$ from S_i , then we may assume that all the vertices that come after $v_{i,k}$ on this path are also in S_i (there is a direct path from $v_{i,k}$ to the outer face in S_i). Therefore, this path is present in $G[A_i]$, and hence in $G(T_i \circ T_j)$, still witnessing that v is in layer L_q . By symmetry, the same holds for vertices in B_j .

Case 2: $G[S_i]$ and $G[S_j]$ are cycles. Assume that $v^* \neq v^\dagger$ in the following; the proof for $v^* = v^\dagger$ is completely analogous. Assume furthermore that $A_i \setminus S_i$ and $B_j \setminus S_j$ are nonempty; otherwise, the claim is trivially fulfilled as the gluing operation degenerates to taking a subgraph of G . Let C_i and C_j denote the cycles induced by S_i and S_j . Both C_i and C_j divide the plane into two regions. Since C_i is an *induced* cycle, the vertices in the unbounded region for C_i can be only from $A_i \setminus B_i$: By Property (vb) of Definition 3.1, we have $L_1 \subseteq A_i$. Moreover, $L_1 \setminus S_i \neq \emptyset$ since L_1 contains at least three vertices. Thus, if this region contains a vertex from $B_i \setminus S_i$, then there is a face containing vertices of $B_i \setminus S_i$ and of $A_i \setminus S_i$. This face is a triangle and thus there is an edge between $B_i \setminus S_i$ and of $A_i \setminus S_i$. This contradicts Property (ii). Thus, all vertices of $B_i \setminus S_i$ are contained in the region enclosed by C_i . By the same argument there, there can be no vertex of $A_i \setminus S_i$ in the region enclosed by C_i . When using the embedding of G for $G[A_i]$, this implies that there is one face such that the vertex set in its boundary is exactly S_i . Similarly, for C_j , the unbounded

region contains all vertices of A_j and no vertices of $B_j \setminus A_j$. This implies in particular that using the embedding of G for $G[B_j]$, the vertex set in the boundary of the outer face is exactly S_j .

Consider now the disjoint union of $G[A_i]$ and $G[B_j]$ where the copies of v^* and v^\dagger introduced by adding $G[A_i]$ and $G[B_j]$ are denoted v_i^* , v_j^* , and v_i^\dagger , v_j^\dagger , respectively. Modify the embedding of $G[B_j]$ so that all vertices of B_j lie in the face whose boundary is S_i . Assume without loss of generality that, in the combinatorial embedding of $G[A_i]$, the face whose boundary is S_i has a counterclockwise face walk $(v_i^*, v_{i,1}, \dots, v_i^\dagger)$ and that the outer face for $G[B_j]$ has a counterclockwise face walk $(v_j^\dagger, v_{j,p}, \dots, v_j^*)$. Herein, observe that the orientation of the face walk is defined by the viewpoint of the face. Thus, the order of the indices is essentially the same for both cycles. If, initially, the order for $G[B_j]$ is the reverse, then we can use the reflection along a vertical line of the original embedding of $G[B_j]$ instead, which reverses the order of the outer face.

Adding the edge $\{v_i^*, v_j^*\}$ to this plane graph can be done without introducing a crossing, yielding a face with face walk $(v_i^*, v_{i,1}, \dots, v_i^\dagger, v_i^*, v_j^*, v_j^\dagger, v_{j,p}, \dots, v_{j,1}, v_j^*)$. Then, adding the edge $\{v_{i,1}, v_{j,1}\}$ can be again done within this embedding without introducing a crossing and such that the resulting face has the face walk $(v_{i,1}, \dots, v_i^\dagger, v_i^*, v_j^*, v_j^\dagger, v_{j,p}, \dots, v_{j,1})$. This process can be continued, that is, we add the edge $\{v_{i,\ell}, v_{j,\ell}\}$ for increasing ℓ , each time obtaining a face in which the edge $\{v_{i,\ell+1}, v_{j,\ell+1}\}$ can be added without separating any vertices with higher index from the face. This is done until, finally, the edge $\{v_i^\dagger, v_j^\dagger\}$ is added. The resulting graph is planar and has an embedding with the same outer face as G in its initial embedding. Contracting each of the edges added between $G[A_i]$ and $G[B_j]$ gives a planar graph. After the contraction, we rename v_i^* to v^* and v_i^\dagger to v^\dagger . By the same arguments as for the path separators, the resulting graph is exactly $G(T_i \circ T_j)$. Thus, $G(T_i \circ T_j)$ is planar and has an embedding such that the boundary of the outer face is L_1 .

It remains to show the r -outerplanarity of $G(T_i \circ T_j)$. The layering of the cycle separator and the fact that v^* and v^\dagger are on the lowest layer that contains vertices from S_i implies that any shortest path from v^* to L_1 and from v^\dagger to L_1 is contained in $G[A_i]$ and thus in $G(T_i \circ T_j)$. Now consider a vertex v from $S_i \setminus \{v^*, v^\dagger\}$. Again the layering implies that there is a shortest path from u to L_1 that contains v^* or v^\dagger and is contained in $G[A_i]$: Starting from v , visit a neighbor of the current vertex that is in S_i and in a lower layer until v^* or v^\dagger is reached, then take the shortest path from this vertex to L_1 (which is contained in $G[A_i]$ by the previous argument). This also implies that there is a shortest path from any vertex $v \in A_i \setminus S_i$ to L_1 that is completely contained in $G[A_i]$: If a shortest path from v to L_1 does not contain vertices from S_i , then this is trivially true. Otherwise, a shortest path from v to L_1 contains a path from v to some vertex u in S_i that is completely contained in $G[A_i]$. Then, concatenating this path with a shortest path from u to L_1 that is completely contained in $G[A_i]$ gives a shortest path from v to L_1 that is in $G[A_i]$. Finally, for each vertex v of $B_j \setminus A_i$, observe first that, since $L_1 \subseteq A_j$, every path from v to L_1 contains a vertex of S_j . Thus, let u denote the first vertex of S_j on a shortest path from v to L_1 . The shortest path from v to u is contained in $G[B_j]$ and there is a shortest path from u to L_1 which is contained in $G[A_i]$. Both subpaths are contained in $G(T_i \circ T_j)$. Thus, the distance of each vertex to a vertex in L_1 is at least as large in G as it is in $G(T_i \circ T_j)$. Hence, $G(T_i \circ T_j)$ is r -outerplanar. \square

4 Application: A problem kernel for Planar Support

We now use the existence of long well-formed separator sequences to give a problem kernel for PLANAR SUPPORT. Assume that the hypergraph has an r -outerplanar support. Observe that, whenever it is convenient, we can assume that this r -outerplanar support is a triangulated disk: triangulating interior faces and adding edges to make L_1 a cycle does not increase the outerplanarity of the graph and also does not destroy the support property. Clearly, we have the desired problem kernel if n can be bounded in terms of m and r . Otherwise, if $m, r \ll n$, then, by Theorem 3.2, there exists a well-formed separator sequence that is long in comparison with m . In this case, intuitively speaking, for at least two separators in this sequence, their “status” must

be the same with respect to the hyperedges of \mathcal{H} crossing them. These two separators can be glued resulting in a new graph. This new graph is not a support for \mathcal{H} since it has less vertices. The missing vertices, however, can be “redrawn” to obtain an r -outerplanar support for \mathcal{H} . We formalize next the concepts discussed above.

Definition 4.1 (Representative support). We call a graph $G = (V, E)$ a *representative support* of a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ if every vertex $u \in D := \mathcal{V} \setminus V$ is covered by some vertex $v \in V$, and G is a support for $\mathcal{H} - D$.

We call an r -outerplanar support of a hypergraph \mathcal{H} a *solution*, and a representative r -outerplanar support a *representative solution* for \mathcal{H} . Using Theorem 3.2, we now show that the size of a smallest representative solution can be upper-bounded by a function of the number m of hyperedges of \mathcal{H} plus the outerplanarity r of a solution. To this end, we first formally define the notion of two separators having the same status with respect to the hyperedges that cross the separators. To simplify the definition, we assume that, in the case of cycle separators, the vertices v^* and v^\dagger also have indices, that is, for all i , if $v^* = v^\dagger$ then we set $v^* := v_{i,p}$ and otherwise $v^* := v_{i,p}$ and $v^\dagger := v_{i,p-1}$.

Definition 4.2 (Separator signature). Let $(A_1, S_1, B_1), \dots, (A_t, S_t, B_t)$ be a well-formed separator sequence of width p of a planar graph $G = (V, E)$ that is a representative support for a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. The *signature* of a separator S_i in this sequence is a triple $(\Gamma_i, \phi_i, \Pi_i)$, where

- $\Gamma_i := \{[u]_{\mathcal{R}} \mid u \in A_i\}$ is the set of twin classes of A_i ,
- $\phi_i : \{1, \dots, p'\} \rightarrow \{[v_{i,j}]_{\mathcal{R}} \mid u \in \mathcal{V}\}, j \mapsto [v_{i,j}]_{\mathcal{R}}$ maps each index of a vertex in S_i to the twin class of that vertex, and
- $\Pi_i := \left\{ (e, j, \ell) \mid \begin{array}{l} e \in \mathcal{E} \wedge j < \ell \wedge v_{i,j}, v_{i,\ell} \in e \wedge v_{i,j} \text{ and } v_{i,\ell} \text{ are in the same connected} \\ \text{component of } G[B_i \cap e] \end{array} \right\}$.

Observe that, in Definition 4.2, G is a representative support for \mathcal{H} , and hence, V does not necessarily contain all vertices of \mathcal{H} . Moreover, the number of distinct separator signatures of a well-formed separator sequence is upper-bounded by a function of p and m : There are at most $2^m - 1$ twin classes in \mathcal{H} . Furthermore, for $i < j$, we have $A_i \subset A_j$, which implies $\Gamma_i \subseteq \Gamma_j$. Thus, either $\Gamma_i = \Gamma_{i+1}$ or Γ_{i+1} has at least one additional twin class. Since the number of twin classes can increase at most $2^m - 2$ times, the number of different Γ_i is less than 2^m . Next, there are at most 2^m choices for a twin class for each $v_{i,j} \in S_i$, leading to at most 2^{mp} different possibilities. For the last part of the signature, we have $m \cdot (p^2 - p)/2$ different triples, and Π_i is an element of the power set of this set of triples. Since $p \leq 2r$, we have the following upper bound on the number of possible signatures:

Observation 4.3. Every well-formed separator sequence of a representative solution has less than $2^{m(r^2+r+1)}$ different separator signatures.

Lemma 4.4. If a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ has a solution, then it has a representative solution with at most $2^{2^{2r(m \cdot (r^2+r+1))} \cdot 6^{2r^2}}$ vertices.

Proof. Let $G = (V, E)$ be a representative solution for \mathcal{H} with the minimum number of vertices, and assume towards a contradiction that $|V| > 2^{2^{2r(m \cdot (r^2+r+1))} \cdot 6^{2r^2}}$. We show that there is a representative support for \mathcal{H} with less than $|V|$ vertices. As mentioned above, we can assume that G is a triangulated disk.

Since G is r -outerplanar with more than $2^{2^{2r(m \cdot (r^2+r+1))} \cdot 6^{2r^2}}$ vertices, by Theorem 3.2, there is a well-formed separator sequence of length at least

$$\left\lfloor \frac{2^r \sqrt{\log 2^{2^{2r(m \cdot (r^2+r+1))} \cdot 6^{2r^2}}}}{6^r} \right\rfloor = \left\lfloor \frac{\sqrt[2r]{2^{2r(m \cdot (r^2+r+1))} \cdot 6^{2r^2}}}{6^r} \right\rfloor = \left\lfloor \frac{2^{m(r^2+r+1)} \cdot 6^r}{6^r} \right\rfloor = 2^{m(r^2+r+1)}.$$

Observation 4.3 and the pigeonhole principle thus imply that there are two separators $T_i = (A_i, S_i, B_i)$ and $T_j = (A_j, S_j, B_j)$, $i < j$, of this sequence that have the same separator signature.

We show that the graph $G(T_i \circ T_j)$ is a representative solution for \mathcal{H} . This will contradict our choice of G , thus proving the claim. First, by Lemma 3.4, $G' := G(T_i \circ T_j)$ is an r -outerplanar graph. Therefore, it remains to show that $G' = (V', E')$ is a representative support.

By Definition 3.3 of the gluing operation, the vertex set of G' is $A_i \cup (B_j \setminus S_j)$ (or equivalently, $(A_i \setminus S_i) \cup B_j$). Since the separators T_i and T_j have the same signature, we have that each twin class of \mathcal{H} with at least one member in G has also at least one member in G' : All vertices that are removed in the gluing operation are from A_j and, since $\Gamma_i = \Gamma_j$, also in A_i . Now, since each vertex of $\mathcal{V} \setminus V$ is covered by some vertex $v \in V$, it follows that each vertex of $\mathcal{V} \setminus V'$ is also covered by some vertex $v' \in V'$. This shows the first of the two properties in Definition 4.1 of representative supports. It remains to show that G' is a support for $\mathcal{H}[V']$.

Consider a hyperedge e' of $\mathcal{H}[V']$. We show that $G'[e']$ is connected. First, let e be a hyperedge of $\mathcal{H}[V]$ such that $e \cap V' = e'$, that is, $e \supseteq e'$ and the vertices of e that are not in e' are all removed during the gluing operation. Observe that such a hyperedge e exists and that, since G is a representative support of \mathcal{H} , $G[e]$ is connected. To show that $G'[e']$ is connected we distinguish two cases.

Case 1: $e \cap S_i = \emptyset$. We either have $e \subseteq A_i \setminus S_i$ or $e \subseteq B_j \setminus S_j$. In both cases, $G[e] = G'[e] = G'[e']$ (as $G[A_i] = G'[A_i]$ and $G[B_j] = G'[B_j]$). Since $G[e]$ is connected, so is $G'[e']$.

Case 2: $e \cap S_i \neq \emptyset$. Observe that $S_i \cap e$ and $S_j \cap e$ are separators in $G[e]$. To show that $G'[e']$ is connected, we show three claims.

Claim 1: In $G'[e']$, each vertex $a \in e' \cap A_i$ is connected to some vertex of $e' \cap S_i$. We have that $G[e]$ is connected, that e contains a vertex of S_i and that $S_i \cap e$ is a separator in $G[e]$. Thus, $G[e]$ contains a path from a to some vertex of S_i that contains only vertices of A_i . Since $G[A_i] = G'[A_i]$ this path is also contained in G' .

Claim 2: In $G'[e']$, each vertex $b \in e' \cap B_j$ is connected to some vertex of $e' \cap S_i$. The claim is trivially true if $b \in S_i$. Thus, assume that $b \in B_j \setminus S_i$. We have that $G[e]$ is connected, that e contains a vertex of S_j , and that $S_j \cap e$ is a separator in $G[e]$. Thus, $G[e]$ contains a path from b to some vertex v of S_j that contains only vertices of B_j . Assume that this path from b to v has minimum length among all paths from b to any vertex in S_j . Let $w \in B_j$ denote the neighbor of v in this path and observe that $w \notin S_j$. Since $e \cap (B_j \setminus S_j) = e' \cap (B_j \setminus S_j)$ and since $G[B_j \setminus S_j] = G'[B_j \setminus S_j]$, this path is also contained in $G'[e' \cap (B_j \setminus S_j)]$. Now let $v := v_{j,k}$, that is, v is the k -th vertex in separator S_j . By Definition 3.3 of the gluing operation, there is in G' an edge from $v_{i,k}$ to w . Observe that $v_{i,k} \in e' \cap S_i$ since $\phi_i = \phi_j$ which implies that $v_{i,k}$ and $v_{j,k}$ are twins. Thus, $G'[e']$ contains a path from u to w to $v_{i,k} \in e' \cap S_i$.

Claim 3: In $G'[e']$, each pair of vertices $u, v \in e' \cap S_i$ is connected. Observe that u and v are connected by a path ($u = p_1, \dots, p_q = v$) in $G[e]$. Since $S_i \cap e$ is a separator in $G[e]$, this path can be decomposed into subpaths that have (respectively) only vertices in $A_i \setminus B_i$, only vertices in $B_i \setminus A_i$, and only vertices in S_i . Let $u = w_1, \dots, w_x = v$ denote the vertices of this path that are in S_i , that is, for each ℓ , $1 \leq \ell < x$, there is in $G[e]$ a path from w_ℓ to $w_{\ell+1}$ that does not contain other vertices from S_i . We show that, in $G'[e']$, there is also such a path. Since each $w_\ell \in e'$, this implies that there is a path from u to v in $G'[e']$.

If w_ℓ and $w_{\ell+1}$ are adjacent in G , then they are also adjacent in G' and, thus, connected in $G'[e']$. Otherwise, if the path from w_ℓ to $w_{\ell+1}$ contains vertices from $A_i \setminus B_i$, then all these vertices are also contained in e' as $A_i \subseteq V'$. Since $G[A_i] = G'[A_i]$, this path is also present in $G'[e']$. In the remaining case, the path contains vertices from $B_i \setminus A_i$. Hence, w_ℓ and $w_{\ell+1}$ are in the same connected component of $G[B_i \cap e]$. Let $v_{i,y} := w_\ell$ and $v_{i,z} := w_{\ell+1}$. Moreover, let $v_{j,y}$ and $v_{j,z}$ denote the vertices that are identified with w_ℓ and $w_{\ell+1}$ in the gluing operation.

Observe that $v_{j,y}$ and $v_{j,z}$ are in the same connected component of $G[B_j \cap e]$ since the separators have the same signature, which implies Π_i^B and Π_j^B . Moreover, observe that $G[B_j]$ is isomorphic to $G'[(B_j \setminus S_j) \cup S_i]$ where the isomorphism maps each vertex of $B_j \setminus S_j$ to itself and maps each vertex of S_i to the vertex of S_j that it is identified with. Consequently, w_ℓ and $w_{\ell+1}$ are in the same connected component of $G'[(B_j \setminus S_j) \cup S_i \cap e']$. Hence, there is a path from w_ℓ to $w_{\ell+1}$ in $G'[e']$. \square

We now use this upper bound on the number of vertices in representative solutions to obtain a problem kernel for PLANAR SUPPORT. First, we show that representative solutions can be extended in a particularly simple way to obtain a solution.

Lemma 4.5. Let $G = (V, E)$ be a representative solution for a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. Then, \mathcal{H} has a solution in which all vertices of $\mathcal{V} \setminus V$ have degree one.

Proof. Assume, without loss of generality, that G is a triangulated r -outerplanar disk. Let G' be the graph obtained from G by making each vertex v of $\mathcal{V} \setminus V$ a degree-one neighbor of a vertex in V that covers v (such a vertex exists by the definition of representative support). Clearly the resulting graph is planar. It is also r -outerplanar: If the neighbor v of a new degree-one vertex is in L_1 , then v can be placed in the outer face. Otherwise, v can be placed in the face whose boundary contains v and a neighbor of v that lies in L_{i-1} (which exists since G is a triangulated disk [3]).

It remains to show that G' is a support for \mathcal{H} . Consider a hyperedge $e \in \mathcal{E}$. Since G is a representative support for \mathcal{H} , we have that $e \cap V$ is nonempty and that $G[e \cap V]$ is connected. In G' , each vertex $u \in e \setminus V$ is adjacent to some vertex $v \in V$ that covers u . This implies, in particular, that $v \in e$. Thus, $G'[e]$ is connected as $G'[e \cap V]$ is connected and all vertices in $e \setminus V$ are neighbors of a vertex in $e \cap V$. \square

We can now use this observation to show that, if there is a twin class that is larger than a minimal representative solution, then we can safely remove one vertex from this twin class.

Lemma 4.6. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph and let $v \in \mathcal{V}$ be a vertex such that $|[v]_{\mathcal{R}}| \geq \alpha$. If \mathcal{H} has a representative solution with less than α vertices, then $\mathcal{H} - v$ has a solution.

Proof. Let $G = (V, E)$ be a representative solution for \mathcal{H} such that $|V| < \alpha$. Then, at least one vertex of $[v]_{\mathcal{R}}$ is not in V and we can assume, without loss of generality, that this vertex is v . Thus, by Lemma 4.5, \mathcal{H} has a support G' in which v has degree one. The graph $G' - v$ is a support for $\mathcal{H} - v$: for each hyperedge e in $\mathcal{H} - v$, we have that $G'[e \setminus \{v\}]$ is connected because v is not a cut-vertex in $G'[e]$ (since it has degree one). \square

Now we combine the observations above with the fact that there are small solutions to obtain a kernelization algorithm.

Theorem 4.7. PLANAR SUPPORT admits a problem kernel with at most $2^m \cdot 2^{2r(m \cdot (r^2 + r + 1)) \cdot 6^{2r^2}}$ vertices which can be computed in linear time. Hence, PLANAR SUPPORT is fixed-parameter tractable with respect to $m + r$.

Proof. Consider an instance $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ of PLANAR SUPPORT and let $v \in \mathcal{V}$ be contained in a twin class of size more than $2^{2r(m \cdot (r^2 + r + 1)) \cdot 6^{2r^2}}$. By Lemma 4.4, if \mathcal{H} has a solution, then it has a representative solution with at most $2^{2r(m \cdot (r^2 + r + 1)) \cdot 6^{2r^2}}$ vertices. By Lemma 4.6, this implies that $\mathcal{H} - v$ has a solution. Moreover, if $\mathcal{H} - v$ has a solution, then this solution is a representative solution for \mathcal{H} . By Lemma 4.5, this implies that \mathcal{H} has a solution. Therefore, \mathcal{H} and $\mathcal{H} - v$ are equivalent instances, and v can be safely removed from \mathcal{H} .

Performing this removal can be done exhaustively in linear time [14]. The removal yields an instance in which each twin class contains at most $2^{2r(m \cdot (r^2 + r + 1)) \cdot 6^{2r^2}}$ vertices; the claimed overall size bound follows since the number of twin classes is at most 2^m . \square

Corollary 4.8. For any fixed $r \in \mathbb{N}$, the problem of deciding whether a given hypergraph \mathcal{H} has an r -outerplanar support is fixed-parameter tractable when parameterized by the number of hyperedges in \mathcal{H} .

5 Constructing well-formed separator sequences

Throughout this section, we assume that G is an r -outerplanar triangulated disk on n vertices. In this section, we prove Theorem 3.2, that is, that G has a well-formed separator sequence of length at least $\lfloor \sqrt[2r]{\log n / 6^r} \rfloor$ and width at most $2r$. The proof is by induction on the outerplanarity r and distinguishes two cases. The first case is when $G - L_1$ contains a “large” block C . In this case, we assume by induction that C has a well-formed separator sequence of a certain length, which we constructively “extend” to a well-formed separator sequence of length t and width at most $2r$ for G (Construction 5.3); we treat this case in Section 5.1. The second case is when there is no large block in $G - L_1$. Then either L_1 is “large”, or the number of blocks in $G - L_1$ is “large”. We give a direct recursive construction that yields in this case a well-formed separator sequence of length at least $\lfloor \sqrt[2r]{\log n / 6^r} \rfloor$ and of width two or three (Construction 5.18); we treat this case in Section 5.2. Observe that the second case includes the base case of outerplanar graphs. Section 5.3 puts all together and proves Theorem 3.2.

5.1 $G - L_1$ contains a large block

Let C denote a block in $G - L_1$. We will show how a well-formed separator sequence for C of length t can be extended into a well-formed separator sequence of length $t/6$ for G . The resulting sequence for G will be either a sequence of induced paths or a sequence of induced cycles. The following terminology will be useful when distinguishing these two possibilities. Let P be an induced path in C such that the two endpoints of P lie on the outermost layer of C . For a vertex v^* in L_1 , we say that v^* is a *cycle-vertex*, or more precisely a *cycle-vertex with respect to P* , if $G[V(P) \cup \{v^*\}]$ is an induced cycle, and for a pair of vertices $\{v^*, v^\dagger\}$ in L_1 , we say that $\{v^*, v^\dagger\}$ is a *cycle-pair* (with respect to P) if v^* and v^\dagger are adjacent, and $G[V(P) \cup \{v^*, v^\dagger\}]$ is an induced cycle.

Lemma 5.1. Let C be a triangulated disk in $G - L_1$. Suppose that C has a well-formed separator sequence $\mathcal{S}' = (A'_1, S'_1, B'_1), \dots, (A'_t, S'_t, B'_t)$, where each S'_i , $i = 1, \dots, t$, is an induced path. Then, there are at most two distinct vertices in L_1 that are cycle-vertices with respect to any path in \mathcal{S}' .

Proof. Proceed by contradiction. Suppose that there exist three distinct vertices $v_1, v_2, v_3 \in L_1$, where v_1 is a cycle-vertex with respect to S'_{i_1} , v_2 is a cycle-vertex with respect to S'_{i_2} , and v_3 is a cycle-vertex with respect to S'_{i_3} . Let $\{u_1, w_1\}$, $\{u_2, w_2\}$, and $\{u_3, w_3\}$ denote the vertices of $S'_{i_1} \cap L_2$, $S'_{i_2} \cap L_2$, and $S'_{i_3} \cap L_2$, respectively. Note that each of these sets indeed consists of two vertices, since, by Property (va) of well-formed separator sequences, each of the three induced paths starts and ends on the outer layer of C , which is L_2 . For the same reason, and because the outer layer of C is a cycle, there is a path P that contains exactly one vertex from each of S'_{i_1} , S'_{i_2} , and S'_{i_3} , possibly some other vertices of L_2 , and only edges that are incident with the outer face of $G - L_1$. Without loss of generality, assume that P contains u_1 , u_2 , and u_3 . Moreover, there is also such a path P' that contains exactly the vertices w_1 , w_2 , and w_3 . Consider the graph that is obtained from G by contracting P and P' . This (multi)graph is a (not necessarily triangulated) r -outerplanar disk with an embedding in which L_1 is the cycle incident with the outer face. Now, let u and w denote the vertices resulting from the path contractions and observe that $u \neq w$. Since u and w are adjacent to v_1 and v_2 , there is cycle C_u containing u , v_1 , and v_2 and only edges with both endpoints in $L_1 \cup \{u\}$. Similarly, there is a cycle C_w containing w , v_1 , and v_2 and only edges with both endpoints in $L_1 \cup \{w\}$. Moreover, these cycles can be chosen so that the regions enclosed by them intersect in v_1 and v_2 . The vertex v_3 is contained in one of these two cycles. If v_3 is contained in C_u , then it cannot be adjacent to w since all its edges are contained in the region enclosed by C_u . Similarly, if v_3 is contained in the cycle C_w , then it cannot be adjacent to u ; a contradiction. \square

Lemma 5.2. Let C be a triangulated disk in $G - L_1$. Suppose that C has a well-formed separator sequence $\mathcal{S}' = (A'_1, S'_1, B'_1), \dots, (A'_t, S'_t, B'_t)$, such that each S'_i , $i = 1, \dots, t$, is an induced path. There can be at most two distinct pairs $\{v_1^*, v_1^\dagger\}$, $\{v_2^*, v_2^\dagger\}$ of vertices such that

- $\{v_1^*, v_1^\dagger, v_2^*, v_2^\dagger\} \subseteq L_1$,
- no vertex of $\{v_1^*, v_1^\dagger, v_2^*, v_2^\dagger\}$ is a cycle-vertex with respect to any path in \mathcal{S}' , and
- $\{v_1^*, v_1^\dagger\}$, $\{v_2^*, v_2^\dagger\}$ are cycle-pairs with respect to any path in \mathcal{S}' .

Proof. Proceed by contradiction, and assume that there exist three distinct pairs $\{v_1^*, v_1^\dagger\}$, $\{v_2^*, v_2^\dagger\}$, $\{v_3^*, v_3^\dagger\}$ in layer L_1 that are cycle-pairs with respect to S'_{i_1} , S'_{i_2} , and S'_{i_3} , respectively. By the linear separation properties of well-formed separator sequences and since C is a triangulated disk in $G - L_1$, there is a path P that contains exactly one vertex from each of S_{i_1} , S_{i_2} , and S_{i_3} , possibly some other vertices of L_2 , and only edges that are incident with the outer face of $G - L_1$. Without loss of generality, assume that P contains u_1 , u_2 , and u_3 . Moreover, there is also such a path P' that contains exactly the vertices w_1 , w_2 , and w_3 . Consider the graph G' that is obtained from G by contracting P and P' . This (multi)graph is an r -outerplanar disk with an embedding in which L_1 is the cycle incident with the outer face. Now, let u and w denote the vertices resulting from the path contractions and observe that $u \neq w$. Assume without loss of generality that v_1^* and v_2^* are adjacent to u and v_1^\dagger and v_2^\dagger are adjacent to w . Since u is adjacent to v_1^* and v_2^* , there is cycle C_u containing u , v_1^* and v_2^* and only edges with both endpoints in $L_1 \cup \{u\}$ (possibly v_1^* and v_2^* ; in this case the cycle is a set of two edges). Similarly, there is a cycle C_w containing w , v_1^\dagger and v_2^\dagger and only edges with both endpoints in $L_1 \cup \{w\}$. Moreover, these cycles can be chosen so that the regions enclosed by them are disjoint. The edge $\{v_3^*, v_3^\dagger\}$ is contained in one of these two cycles since this pair is distinct from the other two cycle-pairs. If $\{v_3^*, v_3^\dagger\}$ is contained in C_u , then neither of its vertices can be adjacent to w . Since v_1^* and v_2^* are not cycle-vertices with respect to \mathcal{S}' no vertex in C_u they are not adjacent to w . Moreover, any other vertex from L_1 in C_u cannot be adjacent to w since all its incident edges are contained in the region enclosed by C_u . Hence, $\{v_3^*, v_3^\dagger\}$ can only be contained in the cycle C_w but then neither endpoint can be adjacent to u ; a contradiction. \square

Using these observations, we can now describe the construction of the desired well-formed separator sequence of length at least $t/6$, where t is the length of the well-formed separator sequence of C . The correctness of the construction is subsequently proven in Lemma 5.4.

Construction 5.3. Let G be an r -outerplanar triangulated disk, where $r > 1$. Suppose that $G - L_1$ has a block C such that C has a well-formed separator sequence $\mathcal{S}' = (A'_1, S'_1, B'_1), \dots, (A'_t, S'_t, B'_t)$. We construct a sequence $\mathcal{S} = (A_1, S_1, B_1), \dots, (A_q, S_q, B_q)$ for G as follows:

Case 1: \mathcal{S}' satisfies Property (vb). That is, each S'_i is a cycle. Then, for $i = 1, \dots, t$, let $S_i := S'_i$, $B_i := B'_i$, $A_i := A'_i \cup L_1$, and define $\mathcal{S} := (A_1, S_1, B_1), \dots, (A_t, S_t, B_t)$.

Case 2: \mathcal{S}' satisfies Property (va). That is, each S'_i is a path. Then we start by partitioning \mathcal{S}' into three subsequences \mathcal{S}'_1 , \mathcal{S}'_2 , and \mathcal{S}'_3 as follows. For each path $S'_i \in \mathcal{S}'$, $i = 1, \dots, t$, if there exists a cycle-vertex with respect to S'_i in L_1 , then add S'_i to \mathcal{S}'_1 ; otherwise, if there exists a cycle-pair with respect to S'_i in L_1 , then add S'_i to \mathcal{S}'_2 . Finally, let $\mathcal{S}'_3 = \mathcal{S}' \setminus (\mathcal{S}'_1 \cup \mathcal{S}'_2)$. To define \mathcal{S} , we distinguish the following cases:

Case 2.1: $|\mathcal{S}'_1| \geq \max\{|\mathcal{S}'_2|, |\mathcal{S}'_3|\}$. Pick a vertex $v^* \in L_1$ that is a cycle-vertex with respect to at least $|\mathcal{S}'_1|/2$ many paths in \mathcal{S}'_1 . Let $\langle S_{i_1}, \dots, S_{i_q} \rangle$ be the sequence of cycles formed by adding v^* to each path in \mathcal{S}'_1 with respect to which v^* is a cycle-vertex, that is, $S_{i_j} = G[V(S'_{i_j}) \cup \{v^*\}]$, where v^* is a cycle-vertex with respect to S'_{i_j} , for $j = 1, \dots, q$; the order of the cycles in the sequence is the order induced by that of the paths in \mathcal{S}'_1 . Each cycle S_{i_j} , $j = 1, \dots, q$, divides the plane into two closed regions $R_{i_j}^1, R_{i_j}^2$ that share S_{i_j} as boundary. Let $R_{i_j}^1$ denote the region that contains L_1 and let $R_{i_1}^1$ be a minimal region in the set of $R_{i_j}^1$'s. Let A_{i_j} be the set of vertices in $R_{i_j}^1$, B_{i_j} be that in $R_{i_j}^2$, and define $\mathcal{S} := (A_{i_1}, S_{i_1}, B_{i_1}), \dots, (A_{i_q}, S_{i_q}, B_{i_q})$.

Case 2.2: $|\mathcal{S}'_2| \geq \max\{|\mathcal{S}'_1|, |\mathcal{S}'_3|\}$. Pick a pair of vertices $\{v^*, v^\dagger\}$ in L_1 that is a cycle-pair with respect to at least $|\mathcal{S}'_2|/2$ many paths in \mathcal{S}'_2 . Let $\langle S_{i_1}, \dots, S_{i_q} \rangle$ be the sequence of cycles formed by adding both v^*, v^\dagger to each path in \mathcal{S}'_2 with respect to which $\{v^*, v^\dagger\}$ is a cycle-pair, that is, $S_{i_j} = G[V(\mathcal{S}'_{i_j}) \cup \{v^*, v^\dagger\}]$, where $\{v^*, v^\dagger\}$ is a cycle-pair with respect to \mathcal{S}'_{i_j} , for $j = 1, \dots, q$. We define \mathcal{S} exactly as we did in Case 2.1 above.

Case 2.3: $|\mathcal{S}'_3| \geq \max\{|\mathcal{S}'_1|, |\mathcal{S}'_2|\}$. Since G is a triangulated disk, we can, for each vertex $v \in L_2$, fix an arbitrary vertex $v' \in N(v) \cap L_1$ [3]. Now, for each $S'_i = (v_{i,1}, \dots, v_{i,p}) \in \mathcal{S}'_3$, considered with respect to its order in \mathcal{S}'_3 , define $S_i = (v'_{i,1}, v_{i,1}, \dots, v_{i,p}, v'_{i,p})$. Let $\langle S_{i_1}, \dots, S_{i_q} \rangle$ be the sequence obtained in this way. Each S_{i_j} , $j = 1, \dots, q$, cuts L_1 into two cycles that enclose two closed regions $R_{i_j}^1$ and $R_{i_j}^2$, where $R_{i_j}^1$ and $R_{i_j}^2$ overlap on S_{i_j} . For $j = 1$, let $R_{i_j}^2$ be the region that contains a vertex $u \notin S_{i_1}$. For each $j > 1$, let $R_{i_j}^1$ be the region that contains a vertex $u \in S_{i_1} \setminus S_{i_2}$. Now, let A_{i_j} be the set of vertices in $R_{i_j}^1$, B_{i_j} be that in $R_{i_j}^2$, and define $\mathcal{S} := (A_{i_1}, S_{i_1}, B_{i_1}), \dots, (A_{i_q}, S_{i_q}, B_{i_q})$.

Lemma 5.4. The sequence \mathcal{S} constructed in Construction 5.3 is a well-formed separator sequence of length at least $t/6$.

Proof. Construction 5.3 defines the sequence \mathcal{S} based on the well-formed separator sequence $\mathcal{S}' := (A'_1, S'_1, B'_1), \dots, (A'_t, S'_t, B'_t)$ of the block C in $G - L_1$ by distinguishing several cases. We show that each of these cases defines a well-formed separator sequence \mathcal{S} for G .

Case 1: \mathcal{S} is constructed according to Case 1 of Construction 5.3. Clearly, Property (i) is satisfied in this case because $V - L_1 = A'_i \cup B'_i$, $i = 1, \dots, t$, and we add L_1 to each A_i . Property (ii) can be seen as follows: By construction, $B_i = B'_i$ and $S_i = S'_i$ encloses B_i . Further, since \mathcal{S}' is a well-formed separator sequence, there is no edge between $A'_i \setminus B'_i$ and $B'_i \setminus A'_i$. Since S_i is itself enclosed within L_1 , there is no edge between $A_i = A'_i \cup L_1$ and $B_i \setminus A_i$. Property (iii) follows for the same reason as Property (ii). Property (iv) is trivially satisfied. Properties (v) to (viii) follow because \mathcal{S}' satisfies them and because the induced cycles in \mathcal{S} are the same as those in \mathcal{S}' , which also implies that \mathcal{S} has length $t > t/6$.

Case 2.1: \mathcal{S} is constructed according to Case 2.1 of Construction 5.3. We first prove the correctness of the construction in this case (i.e., that all the claims made in the construction are correct). The existence of a cycle-vertex v^* with respect to at least $|\mathcal{S}'_1|/2$ paths in \mathcal{S}'_1 follows from: (1) Lemma 5.1, stating that there can be at most two cycle-vertices in L_1 with respect to distinct paths in \mathcal{S}' , and (2) the definition of \mathcal{S}'_1 , which ensures that, for each path in \mathcal{S}'_1 , there exists a cycle-vertex in L_1 with respect to that path.

The statement that all the cycles S_{i_j} , $j = 1, \dots, q$, are nested is true because C is a triangulated disk and \mathcal{S}' is a well-formed separator sequence. Now, by the Jordan curve theorem, each cycle S_{i_j} , $j = 1, \dots, q$, divides the plane into two closed regions $R_{i_j}^1$, $R_{i_j}^2$, one of which is the interior region bounded by the cycle, and the other is the exterior region. Both regions share S_{i_j} as boundary. Since each induced cycle S_{i_j} consists of a path in C plus exactly one vertex in L_1 , which is (i.e., L_1) exterior to C , one of the two closed regions $R_{i_j}^1$ and $R_{i_j}^2$ must contain L_1 . The nestedness of the regions follows from the nestedness of their cycle-boundaries. We now show that \mathcal{S} satisfies all the properties of a well-formed separator sequence.

Property (i) follows trivially from the fact that each vertex is in one of the two regions. Similarly, Property (iii) follows from the fact that only the vertices of \mathcal{S}'_{i_j} are in $R_{i_j}^1$ and $R_{i_j}^2$. Property (ii) follows from the Jordan curve theorem. Now observe that the cycles S_{i_j} , $j = 1, \dots, q$, are nested. This implies that the regions defined by these cycles form two nested sequences as well. Thus, $R_{i_1}^1 \subseteq \dots \subseteq R_{i_q}^1$ and $R_{i_1}^2 \supseteq \dots \supseteq R_{i_q}^2$. Property (iv) now follows from the fact that no two pairs of regions are identical. To show Property (vb), since \mathcal{S} is constructed according to Case 2.1, each $S_i \in \mathcal{S}$ is a cycle of the form $(v^*, v_{i,1}, \dots, v_{i,p}, v^*)$, where $(v_{i,1}, \dots, v_{i,p})$ is an induced path because $S'_i \in \mathcal{S}'$. The vertex v^* on L_1 is adjacent only to vertices in L_2 , and hence only to the

two vertices $v_{i,1}, v_{i,p}$ of S'_i that lie on L_2 . Thus, S_i is an induced cycle. Moreover, $L_1 \subseteq A_1$ follows from the definition of A_1 . Thus, Property (vb) is satisfied. Property (vi) is satisfied since \mathcal{S}' satisfies it. Properties (vii) and (viii) follow because \mathcal{S}' satisfies them, and each separator in \mathcal{S} was obtained from a separator in \mathcal{S}' by adding the same vertex $v^* \in L_1$.

Case 2.2: \mathcal{S} is constructed according to Case 2.2 of Construction 5.3. The existence of a cycle-pair $\{v^*, v^\dagger\}$ with respect to at least $|\mathcal{S}'_2|/2$ paths in \mathcal{S}'_2 follows from:

- (1) Lemma 5.2, stating that there can be at most two cycle-pairs in L_1 with respect to distinct paths in \mathcal{S}' , and
- (2) the definition of \mathcal{S}'_2 , which ensures that, for each path in \mathcal{S}'_2 , there exists a cycle-pair in L_1 with respect to that path.

The correctness of the construction follows by similar arguments to those made in Case 2.1 above. The proof that \mathcal{S} satisfies the properties of a well-formed separator sequence is exactly the same as that for Case 2.1 above, except when arguing that Property (vb) holds, that is, that each cycle S_i in \mathcal{S} is induced. Recall that S_i is obtained by adding two distinct vertices v^* and v^\dagger to an induced path S'_i in \mathcal{S}' . Since S'_i is an induced path, we only need to show that v^* and v^\dagger have exactly one neighbor in S'_i . This is true because all the neighbors of v^* and v^\dagger in C are in L_2 and because $S'_i \in \mathcal{S}'_2$, which implies that the vertices in S'_i are not adjacent to cycle-vertices in L_1 .

Case 2.3: \mathcal{S} is constructed according to Case 2.3 of Construction 5.3. To prove the correctness of the construction, first note that, for each $S'_{i_j} = (v_{i,1}, \dots, v_{i,p}) \in \mathcal{S}'_3$, the two vertices $v'_{i,1} \in L_1 \cap N(v_{i,1})$ and $v'_{i,p} \in L_1 \cap N(v_{i,p})$ in the extended path $S_i = (v'_{i,1}, v_{i,1}, \dots, v_{i,p}, v'_{i,p})$ are distinct and nonadjacent: otherwise, there would be a cycle-vertex or a cycle-pair on L_1 with respect to S'_{i_j} , and hence, S'_{i_j} would belong to \mathcal{S}'_1 or \mathcal{S}'_2 , not to \mathcal{S}'_3 . Thus, each S_{i_j} is a path that lies completely in the closed region of the plane delimited by L_1 that contains C . Therefore, each S_i determines two cycles on L_1 that partition the region of the plane delimited by L_1 and containing C into two closed regions $R_{i_j}^1$ and $R_{i_j}^2$ whose boundaries overlap on S_{i_j} . Let $R_{i_j}^1$ and $R_{i_j}^2$ be the regions as specified in the construction. Then, because \mathcal{S}' is a well-formed separator sequence, and by planarity, they form two nested sequences $R_{i_1}^1 \subseteq \dots \subseteq R_{i_q}^1$ and $R_{i_1}^2 \subseteq \dots \subseteq R_{i_q}^2$, where $R_{i_j}^1$ contains the prefix S_{i_1}, \dots, S_{i_j} of the sequence, and $R_{i_j}^2$ contains the suffix S_{i_j}, \dots, S_{i_q} of the sequence, for $j = 1, \dots, q$.

Now we use these observations to prove that \mathcal{S} satisfies the properties of a well-formed separator sequence. Property (i) follows because every vertex is contained in one of the two regions. Property (iii) follows because S_{i_j} is the only part shared by the two regions. Property (ii) follows from the Jordan curve theorem and planarity. Finally, Property (iv) follows from the nestedness of the two sequences of regions $R_{i_j}^1, j = 1, \dots, q$, and $R_{i_j}^2, j = 1, \dots, q$ mentioned above. To show Property (va), we argue that each path S_i , obtained by adding two distinct vertices $v'_{i,1}$ and $v'_{i,p}$ on L_1 to $S'_i = (v_{i,1}, \dots, v_{i,p})$ is induced. First, observe that S'_i is induced. Second, as observed above, $v'_{i,1}$ and $v'_{i,p}$ have only one neighbor in S'_i (otherwise, there is a cycle-vertex or a cycle pair on L_1 with respect to S'_i , contradicting the placement in \mathcal{S}'_3). Property (vi) and the layering Properties (vii) and (viii) follow because \mathcal{S}' satisfies them, and because each $S_i \in \mathcal{S}$ was obtained from $S'_i \in \mathcal{S}'$ by adding two vertices on L_1 .

It remains to give a lower bound on the length of the well-formed separator sequence generated by Case 2 of Construction 5.3. Observe that one of the sequences $\mathcal{S}'_1, \mathcal{S}'_2$, and \mathcal{S}'_3 generated in Case 2 has length at least $t/3$ since each separator fulfills one of the three Cases 2.1 to 2.3. If this sequence is \mathcal{S}'_3 , then the constructed sequence has length at least $t/3$. In the other two cases, the choice of the cycle-vertex or cycle-pair guarantees that the constructed sequence has length at least $t/6$. \square

5.2 Many vertices in L_1 or many blocks in $G - L_1$

In this case, we first generate a not necessarily well-formed sequence of *separations of order two or three*, from which we later extract a sufficiently long well-formed separator sequence.

Definition 5.5 (Separation). A *separation* of a graph G is a pair (A, B) such that (i) $A \cup B = V(G)$ and (ii) there are no edges between $A \setminus B$ and $B \setminus A$ in G . Informally, we sometimes call A and B the *sides* of (A, B) . The integer $|A \cap B|$ is called the *order* of the separation. We say that a separation (A, B) is *nontrivial* if $A \setminus B \neq \emptyset \neq B \setminus A$.

The construction of the sequence of separations is inductive: We start with an arbitrary trivial separation (A, B) of order two, where A is an edge incident with the outer face and B contains all vertices. With each separation (A, B) , we associate a potential function $q(B)$ that counts the number of vertices of L_1 and the number of blocks remaining on the B -side of the separation:

Definition 5.6 (Potential function). For a vertex set B of G , let

$$q(B) = |B \cap L_1| + b,$$

where b is the number of blocks in $G[B \setminus L_1]$.

Obviously, for our initial separation (A, B) , the value $q(B)$ is “large”. In the following, from a given separation (A, B) of order two or three such that $q(B)$ is “large”, we construct a new separation (A', B') of order two or three such that $q(B') \geq (q(B) - 1)/\ell$ for some small value ℓ . Thus, if the input graph does not have “large” blocks, we obtain a sequence \mathcal{S} of separations of order two or three whose length is roughly logarithmic in the input graph size.

The challenging part is extracting a sufficiently long well-formed separator sequence from \mathcal{S} . We will consider only separations (A, B) of order two or three such that $G[A \cap B]$ contains a path between two vertices in L_1 . If \mathcal{S} contains sufficiently many separations (A, B) such that $G[A \cap B]$ is an edge or an induced path, then we can easily extract a long well-formed separator sequence satisfying Property (va) of Definition 3.1. However, $G[A \cap B]$ might be a triangle. We will show that if \mathcal{S} contains many separations that form triangles with a common edge, then these form nested cycle separators according to Property (vb) of Definition 3.1. Moreover, if \mathcal{S} contains many separators forming triangles without common edges, then we will show that the “bases” of these triangles yield separators of a well-formed separator sequence of width two.

We now formally describe this approach and prove its correctness. First, we formalize the type of separations we are going to generate. These will be candidates for separators in the well-formed separator sequence we are going to create:

Definition 5.7 (Nice separation). A separation (A, B) of order two in a triangulated disk is called *nice* if $A \cap B$ is an edge in G whose endpoints are both in L_1 . A separation (A, B) of order three is called *nice* if there are exactly two vertices in $A \cap B \cap L_1$, the vertex in $(A \cap B) \setminus L_1$ is a common neighbor of the vertices in $A \cap B \cap L_1$, and furthermore, the three-vertex path P in $G[A \cap B]$ with endpoints in L_1 divides the region enclosed by L_1 into a region R_A containing precisely the vertices in A and a region R_B containing precisely the vertices in B .

Note that the regions R_A and R_B are well-defined: $A \cap B \cap L_1$ separates the closed curve C induced by L_1 into two segments C_1, C_2 . Hereby, P does not cross C_1 or C_2 because $V(P) \setminus L_1$ lies in the region enclosed by C . Thus, each segment yields another closed curve when adding the path P .

The following lemma shows that, in triangulated disks, all separators of size two induce exactly two separations (A, B) and (B, A) . This fact will be useful throughout the remainder of this section. Moreover, the lemma shows that if $A \cap B$ is an edge, it divides the region enclosed by L_1 in the same way the path P does for separations of order three in Definition 5.7.

Lemma 5.8. If G has at least four vertices and a nontrivial separation (A, B) of order two, then $G[A \setminus B]$ and $G[B \setminus A]$ are connected components in $G - (A \cap B)$.

Proof. Let (A, B) be a nontrivial separation of order two in G , let $A \cap B = \{u, v\}$, and let $A^- = A \setminus B$ and $B^- = B \setminus A$. Note that $A^- \neq \emptyset$ and $B^- \neq \emptyset$ because (A, B) is nontrivial.

To prove that each of $G[A^-]$ and $G[B^-]$ is connected, it suffices to show that the number of connected components in $G - \{u, v\}$ is at most two.

First we show that v has a neighbor in A^- and one in B^- , both different from u . Since G is a triangulated disk, it is biconnected, whence $G - u$ is connected. Thus, there exists a simple path P in $G - u$ from a vertex in A^- to a vertex in B^- . Since $\{u, v\}$ separates A^- from B^- but u does not, P must contain vertex v . Thus, indeed v has the desired property.

Now suppose towards a contradiction that there are three connected components C_1, C_2, C_3 in $G - \{u, v\}$. At least two of these components, say C_1 and C_2 , must be both in $G[A^-]$ or both in $G[B^-]$; assume, without loss of generality, that C_1 and C_2 are in $G[A^-]$. Again, since G is a triangulated disk, it is biconnected, whence $G - u$ is connected. Thus there is a vertex $z_1 \in C_1$ and a vertex $z_2 \in C_2$ that are neighbors of v . Also, since G is a triangulated disk, there exists a path R containing all the neighbors of v , including u . The graph $R - u$ is composed of two paths P_1 and P_2 . (Note that none of z_1, z_2 is equal to u .) Because z_1 and z_2 are in different connected components of $G - u$, one of them must be in P_1 and the other in P_2 . Since there are no edges between A^- and B^- , all the vertices on P_1 must belong to the same part as z_1 , *i.e.*, to A^- , and all the vertices in P_2 must belong to the same part as z_2 , and, hence, to A^- as well; this contradicts the fact that v has a neighbor in A^- and a neighbor in B^- , both different from u . \square

By definition, nice separations correspond to paths that split the region delimited by L_1 into two closed subregions. It will often be helpful to only argue about these paths, since they, in turn, almost uniquely determine a separation:

Definition 5.9 (Separations induced by paths). Let $u, w \in L_1$ such that $\{u, w\} \in E(G)$. The edge $\{u, w\}$ splits the closed region R of the plane delimited by L_1 into two closed regions R_1, R_2 , whose boundaries overlap on $\{u, w\}$. We say that $\{u, w\}$ *induces* a nice separation (A, B) of order two, where one of its sides (*i.e.*, A or B) consists of the vertices in R_1 and the other side of those in R_2 . Similarly, a (not necessarily induced) path $P := (u, v, w)$ such that $u, w \in L_1$ and $v \notin L_1$, splits R into two regions R_1, R_2 , whose boundaries overlap on P . We say that P *induces* a nice separation (A, B) of order three, one of its sides consists of the vertices in R_1 and the other of those in R_2 .

As we already indicated in the beginning of this section, for nice separations (A, B) of order three, $G[A \cap B]$ might not necessarily be an induced path. Since sequences of such separations obviously do not satisfy Property (va) and do not obviously satisfy Property (vb), it is challenging to construct well-formed separator sequences from long sequences of such *triangular separations*:

Definition 5.10 (Triangular separation). A nice separation (A, B) of order three such that $G[A \cap B]$ is a triangle (*i.e.*, a K_3) is called *triangular* and said to *induce a triangle*. A separation (A, B) is L_1 -*nontrivial* if $(A \setminus B) \cap L_1 \neq \emptyset \neq (B \setminus A) \cap L_1$.

With the next lemma we show that an L_1 -nontrivial triangular separation can be converted into a separation of order two in a unique way. This separation of order two forms an edge; we will call it a “base” of the triangle. This is illustrated in Figure 2. The idea is that if we construct a sufficiently long sequence of triangular separations with mutually distinct bases, then we can construct a well-formed separator sequence out of the sequence of bases.

Note that each nontrivial separation of order two is also L_1 -nontrivial. But this may not be the case for separations of order three if they are triangular and one edge of the triangle is incident with the outer face.

Lemma 5.11. Let (A, B) be a nice triangular separation in G . There is a separation (C, D) of order two such that $C \cap D = A \cap B \cap L_1$, and either $A \subseteq C$ and $B \cap L_1 \subseteq D$ or $B \subseteq D$ and $A \cap L_1 \subseteq C$. Moreover, if (A, B) is L_1 -nontrivial, then (C, D) is unique.

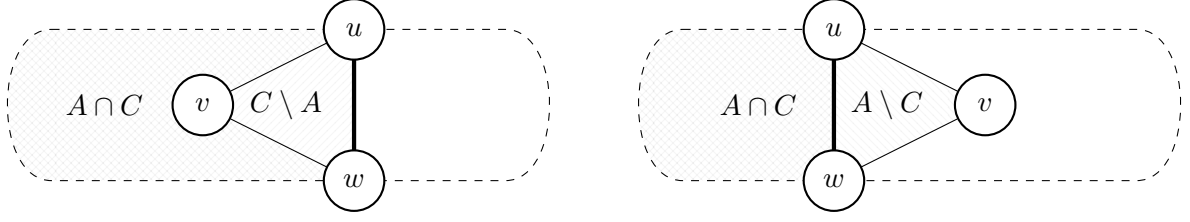


Figure 2: Two triangular separations (A, B) . In both pictures, the dashed line is layer L_1 , part A of the triangular separations (A, B) is hatched in a north west pattern. For each separation, the separation (C, D) of order two as in Lemma 5.11 is shown, where part C is hatched in a north east pattern. The edge $C \cap D$ is drawn in bold.

Proof. Let $S = A \cap B \cap L_1$. Since (A, B) is a triangular separation, $S \subseteq L_1$ is an edge in G , splitting the closed region R delimited by L_1 into two regions R_1, R_2 . Fix R_1 to be those region that contains the middle vertex of the path P that induces (A, B) (note that not both regions can contain the middle vertex). There are two separations induced by S : $(V(R_1), V(R_2))$ and $(V(R_2), V(R_1))$, where $V(R)$ denotes the set of vertices contained in region R . We claim that one of these separations fulfills the conditions of the lemma.

The three-vertex path P with endpoints in S separates R into a region R_A containing A and a region R_B containing B . Since P cannot cross S and since the middle vertex of P is in R_1 , at least one of R_A or R_B is contained in R_1 . If R_A is contained in R_1 , then we take $(C, D) := (V(R_1), V(R_2))$. Analogously, if R_B is in R_1 , then we take $(C, D) := (V(R_2), V(R_1))$. Clearly, (C, D) fulfills the condition that $A \subseteq C$ or $B \subseteq D$. To see that in the first case also $B \cap L_1 \subseteq D$, observe that the boundary of R_B differs from the boundary of R_2 only in P . Since P intersects L_1 in the same points as S , the region R_B cannot enclose more vertices of L_1 than R_2 . The proof for showing that if $B \subseteq D$ then $A \cap L_1 \subseteq C$ is analogous. Hence, (C, D) exists as claimed.

It remains to show uniqueness in the case when (A, B) is L_1 -nontrivial. To see this, note that ambiguity in the definition of (C, D) can only occur if both R_A and R_B are contained in R_1 . This is impossible, however: Because (A, B) is L_1 -nontrivial, each of $A \setminus B$ and $B \setminus A$ contains a vertex of L_1 . One of these vertices is in R_1 , while the other is in R_2 . \square

Definition 5.12 (Base of a triangular separation). For a nice, triangular separation (A, B) , we call a separation (C, D) as in Lemma 5.11 a *base* of (A, B) . If, in addition, (A, B) is L_1 -nontrivial, we say that (A, B) *points left* if $A \subseteq C$ and that it *points right* otherwise.

Note that L_1 -trivial triangular separations (A, B) have both $(V(G), A \cap B \cap L_1)$ and $(A \cap B \cap L_1, V(G))$ as bases. Moreover, note that the separation (A, B) shown in the left picture of Figure 2 points left, whereas the separation in the right picture points right.

Inductive construction of a large sequence of nice separations. We now show how to construct a large family of nice separations, from which a long well-formed separator sequence will be extracted. That is, as described in the outline of the approach, given a separation (A, B) , we want to construct a new separation (A', B') such that the potential function fulfills $q(B') \geq (q(B) - 1)/\ell$ for some small number ℓ . The blocks play a crucial role when defining the new separation; we consider them first. The proof of the following lemma is illustrated in Figure 3.

Lemma 5.13. Let (A, B) be a nice separation of order three for G , where $A \cap B = \{u, v, w\}$ with $v \notin L_1$. Suppose that there is a block C in $G[B \setminus L_1]$ containing a triangle $\{v, x_1, x_2\}$. Then, there is a nice separation (A', B') of order three for G , where $A' \cap B' = \{u', v', w'\}$, $u', w' \in L_1$, and $v' \in C$, satisfying $A \subsetneq A'$, $B \supsetneq B'$, and $q(B') \geq (q(B) - 1)/|C|$.

Proof. The path (u, v, w) splits the region of the plane delimited by the outermost layer L_1 into two closed regions, one containing A and the other containing B , whose boundaries overlap

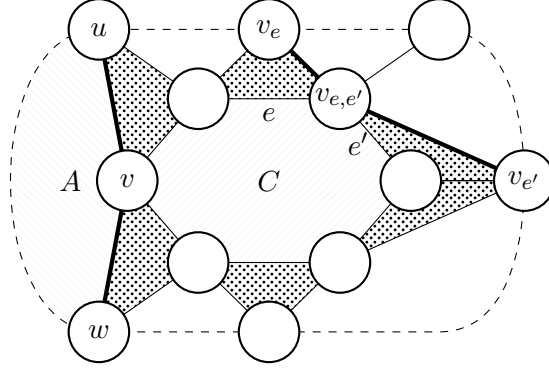


Figure 3: Illustration for the proof of Lemma 5.13. The dashed line is layer L_1 . A separation (A, B) is shown: the A -part is hatched in north west lines and the path $A \cap B = \{u, v, w\}$ is shown in bold. A block C of $G - L_1$ is shown hatched in north east lines. Each outer edge of C is incident with a dotted triangle: these triangles are devoid of vertices of G . For two outer edges e, e' of C , the path $(v_e, v_{e,e'}, v_{e'})$, which induces another separation, is shown in bold.

on u, v, w ; let R be the region of the two that contains B . Since G is a triangulated disk, so is R . Let γ be the boundary cycle of R formed by u, v, w and one of the two paths between u and w on L_1 , and note that every vertex in $B \cap L_1$ is on γ . Since C is a block in $G[B \setminus L_1]$ and G is a triangulated disk, C is a triangulated disk as well. Therefore, the outermost layer γ_C of C is a cycle containing v . Since C is a block and R is triangulated, it follows from the maximality of C that, for each edge e of γ_C , there is a vertex $v_e \in \gamma$ such that v_e forms a triangle with e (i.e., v_e is adjacent to both endpoints of e) whose interior is devoid of vertices of G . Any two consecutive edges e and e' on γ_C such that $v_e \neq v_{e'}$ define a nice separation $(A_{e,e'}, B_{e,e'})$ for G of order three. It is induced by the (not necessarily induced) path $(v_e, v_{e,e'}, v_{e'})$, where $v_{e,e'} \in \gamma_C$ is the common endpoint of the two consecutive edges e and e' . (This is true because $(v_e, v_{e,e'}, v_{e'})$ is a path between two vertices on L_1 that contains a vertex not in L_1 .) In the separation $(A_{e,e'}, B_{e,e'})$, we designate $A_{e,e'}$ to be the side of the separation that is delimited by the path $(v_e, v_{e,e'}, v_{e'})$ and containing A , and $B_{e,e'}$ to be the other side, which is contained in B . Clearly, for $v_{e,e'} \neq v$, we have $A \subsetneq A_{e,e'}$ and $B_{e,e'} \subsetneq B$. Now we go around γ_C defining the separations $(A_{e,e'}, B_{e,e'})$ for each two consecutive edges e, e' on γ_C such that $v_{e,e'} \neq v$. The vertices v_e , where $e \in \gamma_C$, belong to γ , and every vertex in $\gamma \setminus \{v\}$ is either equal to one of the v_e 's or is situated between two of them on γ . Therefore, every vertex in $B \cap L_1$ belongs to $B_{e,e'}$ for some separation $(A_{e,e'}, B_{e,e'})$. Moreover, because γ_C is a cycle inside the cycle γ , it is easy to verify that each block in $G[B \setminus L_1]$ other than C must belong to $B_{e,e'}$ for some separation $(A_{e,e'}, B_{e,e'})$ defined in the above process. Let (A', B') be the nice separation among all the $(A_{e,e'}, B_{e,e'})$ that maximizes the value $q(B_{e,e'})$. From the above discussion, it follows that $A \subsetneq A'$, $B \supsetneq B'$, and $q(B') \geq (q(B) - 1)/|C|$ (the minus 1 is to account for C). \square

We now use Lemma 5.13 in the inductive construction of nice separations.

Lemma 5.14. Let (A, B) be a nice separation in G and ℓ be the maximum of the number 2 and the size of a largest block in $G - L_1$. If $q(B) \geq \ell$, then there is a nice separation (A', B') such that:

1. $A \subseteq A'$, $B \supseteq B'$;
2. $q(B') \geq (q(B) - 1)/\ell$; and
3. if $A = A'$ or $B = B'$, then (A, B) and (A', B') are of different order.

Proof. We distinguish between the cases of (A, B) having order two or three.

Case 1: (A, B) is a separation of order two. Let $A \cap B = \{u, v\}$. Since (A, B) is a nice separation, $\{u, v\}$ is an edge in G . Since $q(B) \geq \ell > 2$, there is at least one vertex in $B \setminus \{u, v\}$,

and hence, there is an inner face F with $V(F) \subseteq B$ that is incident with $\{u, v\}$ and that contains a vertex $w \notin \{u, v\}$.

Case 1.1: $w \in L_1$. Each of the two edges $\{u, w\}$ and $\{v, w\}$ is between two vertices on L_1 , and hence a separator for G . Thus, by Lemma 5.8 $\{u, w\}$ induces a unique nice separation (A_1, B_1) of order two such that both u and v are in A_1 , and $\{v, w\}$ induces a unique nice separation (A_2, B_2) of order two such that both u and v are in A_2 . Let (A', B') be the separation out of (A_1, B_1) and (A_2, B_2) that maximizes $q(B')$. Since (A, B) is a nice separation of G such that $A \cap B = \{u, v\}$, and since A' is the side of G that contains u and v , it follows from the definition of A' that $A \subseteq A'$. Since $A' \cap B'$ is a separator contained in B , it also follows that $B \supseteq B'$. Now, the separation (A', B') was chosen to maximize $q(B')$. Thus, $q(B') \geq q(B)/2 \geq (q(B) - 1)/\ell$ (because $q(B)$ is basically split between $q(B_1)$ and $q(B_2)$). Since $A' \neq A$ and $B' \neq B$, Condition 3 of the lemma is clearly satisfied by our choice of A' and B' .

Case 1.2: $w \notin L_1$. Let $A' := A \cup \{w\}$ and $B' := B$. Since (A, B) is a separation, clearly so is (A', B') . Moreover, since $w \notin L_1$, (A', B') is a nice separation. Clearly, Condition 1 is fulfilled by (A', B') . Condition 2 is fulfilled because $B' = B$, and hence $q(B) = q(B') \geq (q(B) - 1)/\ell$. Finally, (A, B) and (A', B') are clearly of different order, implying that Condition 3 holds.

Case 2: (A, B) is a separation of order three. Let $A \cap B = \{u, v, w\}$. By the definition of nice separation, there exists a vertex $v \in (A \cap B) \setminus L_1$. We distinguish whether v has none, one, or multiple neighbors in $B \setminus A$.

Case 2.1: v has no neighbors in $B \setminus A$. Since v has no neighbors in $B \setminus A$, (A, B) induces an empty triangle $\{u, v, w\}$. Let (A', B') be the unique base (see Lemma 5.11) of (A, B) such that $A' \supseteq A$. Condition 1 holds. Condition 3 holds because (A, B) has order three and (A', B') has order two. Finally, since $v \notin L_1$, we have $|B \cap L_1| = |B' \cap L_1|$. Moreover, the number of blocks in $G[B' \setminus L_1]$ is at least that in $G[B \setminus L_1]$ minus one. Therefore, $q(B') \geq q(B) - 1 \geq (q(B) - 1)/\ell$, where the last inequality is true because $\ell \geq 2$.

Case 2.2: v has a neighbor $x \in (B \setminus A) \cap L_1$. It is easy to see that, in this case, each of the two paths (u, v, x) and (x, v, w) induces a nice separation of order three. Let (A_1, B_1) be the separation induced by (u, v, x) , where A_1 is the side containing w , and (A_2, B_2) that induced by (x, v, w) , where A_2 is the side containing u . Note that $B_1 \subsetneq B$ (proper containment because $w \notin B_1$) and $B_2 \subsetneq B$ (proper containment because $u \notin B_2$). Moreover, we have $A \subsetneq A_1$ and $A \subsetneq A_2$. Let (A', B') be the separation out of (A_1, B_1) and (A_2, B_2) maximizing $q(B')$. Since $V(B) = V(B_1) \cup V(B_2)$, it is easy to see that $q(B') \geq q(B)/2 \geq (q(B) - 1)/\ell$. The above shows that Conditions 1 and 2 hold. Moreover, since the inclusions are proper, Condition 3 is satisfied.

Case 2.3: v has exactly one neighbor $x \in B \setminus A$, which is not in L_1 . Since G is a triangulated disk, x is a common neighbor of u and w . Moreover, the interior of the triangles (u, x, v) and (v, x, w) must be devoid of vertices of G . Since $u, w \in L_1$ and $x \notin L_1$, (u, x, w) induces a nice separation (A', B') of order three, where $A' = A \cup \{x\}$ and $B' = B \setminus \{v\}$. Clearly Condition 1 is met. Moreover, since $B' = B \setminus \{v\}$, and $v \notin L_1$, $q(B') \geq q(B) - 1 \geq (q(B) - 1)/\ell$ (the first inequality is true because the number of blocks in $G[B' \setminus L_1]$ is at least that in $G[B \setminus L_1]$ minus one), and Condition 2 is met. Because neither $A = A'$ nor $B = B'$, Condition 3 holds.

Case 2.4: v has at least two neighbors in $B \setminus A$ that are not in L_1 . Since G is a triangulated disk, two neighbors $x_1, x_2 \in B \setminus A$ of v are adjacent, and hence, v, x_1, x_2 are part of a block C in $G[B \setminus L_1]$. Therefore, the preconditions of Lemma 5.13 are met, and there is a separation (A', B') satisfying Conditions 1 and 2. Since we have $A' \subsetneq A$ and $B \supsetneq B'$, Condition 3 holds. \square

Extracting a well-formed separator sequence. By successively applying Lemma 5.14, we can generate a long sequence of nice separations, given that our input graph is sufficiently large. It remains to extract a long well-formed separator sequence from the long sequence of nice separations. As mentioned before, we have to be careful when using nice separations (A, B) for which

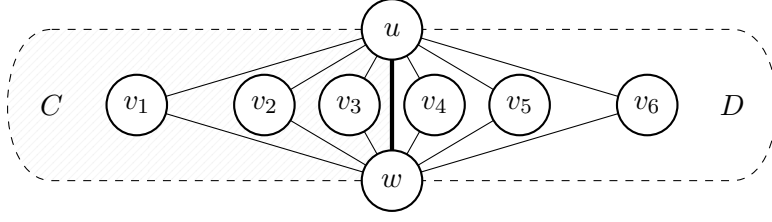


Figure 4: A hinged sequence of triangular separations induced by the triangles $((u, v_i, w))_{1 \leq i \leq 6}$ with their common edge drawn boldly. All separations have a common base (C, D) , whose C -part is hatched. The sequence decomposes into a maximal homogeneous subsequence $((u, v_i, w))_{1 \leq i \leq 3}$ pointing left and a maximal homogeneous subsequence $((u, v_i, w))_{4 \leq i \leq 6}$ pointing right. Note that each homogeneous subsequence satisfies Property (vb) of well-formed separator sequences if we set $v^* = u$ and $v^\dagger = w$.

$G[A \cap B]$ is a triangle, since long sequences of triangles do not immediately fit into Definition 3.1 of well-formed separator sequences. In Lemma 5.11, we have already seen that L_1 -nontrivial triangular separations can uniquely be mapped to nice separations of order two—their bases. If the sequence of bases of triangular separations contains many mutually distinct bases, we will construct a well-formed separator sequence from the bases. If not, then a long sequence of triangular separations will contain many triangles with a common base. This is captured in the following definition and lemma and illustrated in Figure 4.

Definition 5.15 (Linear, hinged, and homogenous sequences). We extend the definitions of triangular, L_1 -nontrivial, and pointing left or right, to sequences of separations in a natural way: For some property $\Pi \in \{\text{linear, hinged, homogeneous}\}$, a sequence \mathcal{S} is Π (*i.e.*, satisfies Π) if each separation in \mathcal{S} is Π . Moreover, a sequence \mathcal{S} of separations is

- linear* if, for each pair $(A, B), (A', B')$ of consecutive separations in \mathcal{S} , we have $A \subsetneq A'$ and $B \supsetneq B'$;
- hinged* if it is triangular and if, for each pair $(A, B), (A', B')$ of separations in \mathcal{S} , we have $A \cap B \cap L_1 = A' \cap B' \cap L_1$; and
- homogeneous* if it is hinged and either points left or points right (in particular, \mathcal{S} is L_1 -nontrivial and triangular).

When two triangular separations have the same edge between two vertices in L_1 , then they have a common base:

Lemma 5.16. Let (A, B) and (A', B') be two triangular separations such that $A \cap B \cap L_1 = A' \cap B' \cap L_1$ and $A \subsetneq A'$ and $B \supsetneq B'$. Each base of (A, B) is also a base of (A', B') .

Proof. Let (C, D) be a base of (A, B) . We prove that (C, D) is a base of (A', B') . We distinguish whether (C, D) is a trivial separation or not.

Case 1: (C, D) is a trivial separation. Since (A, B) is triangular, $C \cap D$ is an edge in G . Since $C \subseteq D$ or $D \subseteq C$, the edge $C \cap D$ is incident with the outer face. Hence, each path through G with endpoints in $C \cap D$, containing the edge $C \cap D$, and using otherwise only vertices not in L_1 , encloses a region that contains $C \cap D$ as the only vertices in L_1 . Hence, by the definition of nice separation, $A' \cap L_1 \subseteq C \cap D$ or $B' \cap L_1 \subseteq C \cap D$. Since clearly both $B', A' \subseteq C \cup D$, the separation (C, D) is a base of (A', B') by definition.

Case 2: (C, D) is a nontrivial separation. Clearly, (A', B') has a base (C', D') . Moreover, $C \cap D = C' \cap D'$ since $A \cap B \cap L_1 = A' \cap B' \cap L_1$. Thus, since (C, D) is nontrivial, so is (C', D') , which implies that (A', B') is L_1 -nontrivial. Therefore, (C', D') is unique by Lemma 5.11. We prove that $(C, D) = (C', D')$. Assume for the sake of contradiction that $(C, D) \neq (C', D')$. Since $C \setminus D$ and $D \setminus C$ are connected components in $G - (C \cap D)$ by Lemma 5.8, we have $C \setminus D = D' \setminus C'$

and $D \setminus C = C' \setminus D'$, meaning that $C = D'$ and $C' = D$. If both (A, B) and (A', B') point left, then $A \subseteq C = D'$ and $A \subseteq C'$; a contradiction since (A, B) is nontrivial and, therefore, $|A| > 2$. Similarly, (A, B) and (A', B') cannot both point right. Assume now that (A, B) points left and (A', B') points right. Since (A, B) points left, $A \subseteq C$ and, since it is L_1 -nontrivial, this implies $A \cap L_1 \cap (C \setminus D) \neq \emptyset$. Thus, since $A \subsetneq A'$, one has $A' \cap L_1 \cap (D' \setminus C') \neq \emptyset$. However, since (A', B') points right, one gets $A' \cap L_1 \subseteq C'$, a contradiction. The case that (A, B) points right and (A', B') points left is analogous. Thus, $(C, D) = (C', D')$. \square

In particular, if (A, B) and (A', B') are L_1 -nontrivial, then they share a unique base. Moreover, Lemma 5.16 extends to hinged sequences.

Corollary 5.17. Let \mathcal{S} be a linear hinged sequence of triangular separations. A base of one separation in \mathcal{S} is a base of each separation in \mathcal{S} .

Thus, we may speak of the base of a linear hinged sequence of triangular separations.

We will construct a well-formed separator sequence from a long sequence of nice separations as follows: if the sequence contains many triangular separations, then either we use their bases as separators if there are enough mutually distinct bases, or use a linear, hinged, homogeneous sequence as a well-formed separator sequence of the cycle type (satisfying Property (vb) of Definition 3.1). If the sequence does not contain many triangular separations, we simply throw them away. Formally, the construction of the well-formed separator sequence is as follows.

Construction 5.18. Let G be an r -outerplanar triangulated disk and $t \in \mathbb{N}$. We construct a well-formed separator sequence \mathcal{T} of width two or three and length t for G . Let A_1 be any edge incident with the outer face of G and let $B_1 = V(G)$. Clearly, (A_1, B_1) is a nice separation of order two. Set $i := 1$; while Lemma 5.14 is applicable to separation (A_i, B_i) , let (A_{i+1}, B_{i+1}) be the resulting (nice) separation from the application of the lemma, and set $i := i + 1$. Let \mathcal{S} be the sequence of all the separations (A_i, B_i) defined by the above iterative process. We distinguish the following cases:

Case 1: There is a homogeneous subsequence \mathcal{S}' of length at least t in \mathcal{S} . Pick a base (C, D) of a separation in \mathcal{S}' and define a sequence \mathcal{T} as follows: If \mathcal{S}' points left, then $\mathcal{T} := ((A \cup (D \setminus C), A \cap B, B \setminus (D \setminus C)))_{(A,B) \in \mathcal{S}'}$, inheriting the order from \mathcal{S}' . Otherwise, $\mathcal{T} := ((B \cup (C \setminus D), A \cap B, A \setminus (C \setminus D)))_{(A,B) \in \mathcal{S}'_{\text{rev}}}$, where $\mathcal{S}'_{\text{rev}}$ is the sequence \mathcal{S}' in reverse order.

Case 2: There is an L_1 -trivial, hinged subsequence \mathcal{S}' of length at least t in \mathcal{S} . Let (A', B') be the first separation in \mathcal{S}' and define a sequence \mathcal{T} as follows: If $L_1 \subseteq A'$ then $\mathcal{T} := ((A, A \cap B, B))_{(A,B) \in \mathcal{S}'}$, inheriting the order from \mathcal{S}' . Otherwise, $\mathcal{T} := ((B, A \cap B, A))_{(A,B) \in \mathcal{S}'_{\text{rev}}}$, where $\mathcal{S}'_{\text{rev}}$ is the sequence \mathcal{S}' in reverse order.

Case 3: There are at least $2t$ maximal homogeneous subsequences of \mathcal{S} . Take the sequence of their bases, inheriting the order from \mathcal{S} , and remove duplicates. Based on the resulting sequence \mathcal{S}' of bases, define the sequence $\mathcal{T} := ((C, C \cap D, D))_{(C,D) \in \mathcal{S}'}$ inheriting its order from \mathcal{S}' .

Case 4: None of the above. Remove each triangular separation from \mathcal{S} . Let \mathcal{S}' be the subsequence of \mathcal{S} containing only separations of order two, or only of order three, whichever is largest. Define the sequence $\mathcal{T} := ((A, A \cap B, B))_{(A,B) \in \mathcal{S}'}$ inheriting its order from \mathcal{S}' .

We next prove that the sequence \mathcal{T} constructed above has length at least t , regardless of the case according to which it was constructed. To this end, we have to prove that Case 3 does not discard too many duplicate bases. We will rely on the following lemma.

Lemma 5.19. Let \mathcal{P} and \mathcal{P}' be two maximal homogeneous subsequences of a linear, triangular sequence \mathcal{R} of nice separations such that each separation of \mathcal{P} comes before each separation of \mathcal{P}' in \mathcal{R} . If the base of \mathcal{P} is also the base of \mathcal{P}' , then \mathcal{P} points left and \mathcal{P}' points right. Moreover, in that case, there is no separation in \mathcal{R} between any pair of separations in \mathcal{P} and \mathcal{P}' .

Proof. Let (A, B) in \mathcal{P} and (A', B') in \mathcal{P}' and let (C, D) be their base. We first show the lemma in the case when there is no separation between \mathcal{P} and \mathcal{P}' in \mathcal{R} and then show that there cannot be separations in between.

Case 1: There is no separation in \mathcal{R} between \mathcal{P} and \mathcal{P}' . Since \mathcal{P} and \mathcal{P}' are maximal homogeneous subsequences, and there is no separation in between, they point into different directions. Assume for the sake of contradiction that \mathcal{P} points right and \mathcal{P}' points left. By Definition 5.12, that means $B \subseteq D$. Moreover, since $(A \cap B) \setminus L_1$ contains at least one vertex, we have $(D \setminus C) \cap A \neq \emptyset$ by Lemma 5.11. However, from $A \subsetneq A'$ it then follows that $A' \setminus C \neq \emptyset$, which is a contradiction to $A' \subseteq C$ since (A', B') points left. Hence, if there is no separation between \mathcal{P} and \mathcal{P}' in \mathcal{R} , then \mathcal{P} points left and \mathcal{P}' points right.

Case 2: There is a separation (\hat{A}, \hat{B}) between \mathcal{P} and \mathcal{P}' in \mathcal{R} . We first show that (C, D) is the base of each such separation (\hat{A}, \hat{B}) . Since concatenating \mathcal{P} and \mathcal{P}' yields a linear triangular sequence of nice separations with no separations between \mathcal{P} and \mathcal{P}' , Case 1 shows that \mathcal{P} points left and \mathcal{P}' points right. Without loss of generality (due to symmetry) assume that (\hat{A}, \hat{B}) points left. To prove that (C, D) is the base of (\hat{A}, \hat{B}) , by Corollary 5.17, it suffices to prove that appending (\hat{A}, \hat{B}) to \mathcal{P} yields a homogeneous sequence, that is, $\hat{A} \cap \hat{B} \cap L_1 = A \cap B \cap L_1$. Note that $C \cap D = A \cap B \cap L_1 = A' \cap B' \cap L_1$. Since \mathcal{R} is linear, $A \subsetneq \hat{A}$ and $\hat{B} \supsetneq B'$, which implies $C \cap D \subseteq \hat{A} \cap \hat{B} \cap L_1$. Even equality holds since (\hat{A}, \hat{B}) is nice. Thus, appending (\hat{A}, \hat{B}) to \mathcal{P} yields a homogeneous sequence, implying that (C, D) is the base of (\hat{A}, \hat{B}) by Corollary 5.17. We infer that (C, D) is the base of each separation in \mathcal{R} between \mathcal{P} and \mathcal{P}' .

Now, assume, towards a contradiction, that there are separations between \mathcal{P} and \mathcal{P}' in \mathcal{R} . Since \mathcal{P} and \mathcal{P}' are maximal, there is a maximal triangular homogeneous subsequence $\hat{\mathcal{P}}$ succeeding \mathcal{P} in \mathcal{R} and there is a maximal triangular homogeneous subsequence $\hat{\mathcal{P}}'$ preceding \mathcal{P}' in \mathcal{R} . By the choice of \mathcal{P} and \mathcal{P}' , both these sequences are nonempty. By Case 1, $\hat{\mathcal{P}}$ points right and $\hat{\mathcal{P}}'$ points left. However, concatenating $\hat{\mathcal{P}}'$ and $\hat{\mathcal{P}}$ yields a sequence that is linear, triangular, has the same base as \mathcal{P} and \mathcal{P}' , and no separations between $\hat{\mathcal{P}}'$ and $\hat{\mathcal{P}}$. Thus, Case 1 is applicable to this sequence, which leads to a contradiction since then, by Case 1, $\hat{\mathcal{P}}$ points left and $\hat{\mathcal{P}}'$ points right. \square

Furthermore, we need to prove that, after removing the triangular separations in Case 4, there still remain sufficiently many separations. For this, we need the following lemmas.

Lemma 5.20. Each linear, hinged sequence of triangular separations consists of homogeneous subsequences or is L_1 -trivial.

Proof. We prove that a linear, hinged sequence of triangular separations \mathcal{S}' that does not consist of homogeneous subsequences is L_1 -trivial. Note that \mathcal{S}' contains a L_1 -trivial separation (A, B) as, otherwise, (A, B) either points left or right by Definition 5.12 and is thus part of a homogeneous subsequence. Therefore, (A, B) has *two* trivial bases. Furthermore, by Corollary 5.17, *both* bases of (A, B) are bases of \mathcal{S}' . This implies that each separation in \mathcal{S}' has two bases and is, by Lemma 5.11, L_1 -trivial. \square

Lemma 5.21. There are at most two maximal subsequences of the linear sequence \mathcal{S} in Construction 5.18 that are both L_1 -trivial and hinged.

Proof. Assume that there are three subsequences of \mathcal{S} as above. Pick a separation (A_1, B_1) , (A_2, B_2) , (A_3, B_3) out of each of them. A maximal hinged subsequence is consecutive in \mathcal{S} , whence we may assume $A_1 \subsetneq A_2 \subsetneq A_3$ and $B_1 \supsetneq B_2 \supsetneq B_3$ without loss of generality. Furthermore, $A_1 \cap B_1 \cap L_1 \neq A_2 \cap B_2 \cap L_1 \neq A_3 \cap B_3 \cap L_1$, since \mathcal{S} is linear and by the maximality of the subsequences. By Lemma 5.20, each of the three sequences is L_1 -trivial. Thus, $A_i \cap B_i \cap L_1$, $i \in \{1, 2, 3\}$, is an edge incident with the outer face. Thus, there are two vertices $u, v \in A_1 \cap B_1$, not necessarily distinct, such that $u \in A_2 \setminus B_2$ and $v \in A_3 \setminus B_3$. Let P_2 be the path inducing (A_2, B_2) and denote the corresponding regions by R_2^A, R_2^B , which enclose A_2 and B_2 , respectively. Analogously, let P_3 be the path inducing (A_3, B_3) and R_3^A, R_3^B be the corresponding

regions. Since P_2 and P_3 have length three, they do not cross each other. Since R_2^A contains u , and R_3^A contains v , this means that R_2^A contains R_3^B . Since $B_3 \supsetneq B_2$, we have $A_2 = V$. This is a contradiction to the fact that $A_2 \subsetneq A_3$. \square

We are ready to prove a lower bound on the length of the separator sequence \mathcal{T} generated in Construction 5.18.

Lemma 5.22. Let $\ell \geq 2$ be an upper bound on the size of each block in $G - L_1$ and $k > 0$ be a lower bound on $q(V(G))$. We can carry out Construction 5.18 in such a way that it yields a sequence of length at least $\sqrt{(\log_\ell k + 1)/2} - 1$.

Proof. Let us first find a lower bound on the length i_m of the initial sequence $\mathcal{S} = ((A_i, B_i))_{1 \leq i \leq i_m}$ that Construction 5.18 generates using Lemma 5.14. For $i < i_m$, we have $q(B_{i+1}) \geq (q(B_i) - 1)/\ell$ by Condition 2 of Lemma 5.14. It is not hard to check that i_m is at least the largest integer fulfilling

$$\ell \leq \frac{q(B_1)}{\ell^{i_m-1}} - \sum_{i=1}^{i_m-1} \frac{1}{\ell^i},$$

which is satisfied for all i_m that satisfy

$$\ell \leq \frac{k}{\ell^{i_m-1}} - \frac{1 - 1/\ell^{i_m}}{1 - 1/\ell} + 1.$$

We claim that $i_m \geq \log_\ell k - 1$. Indeed, substituting this term for i_m , we obtain

$$\begin{aligned} \ell - 1 &\leq \frac{k}{\ell^{\log_\ell k - 2}} - \frac{1 - 1/\ell^{\log_\ell k - 1}}{1 - 1/\ell}, \\ \ell - 1 &\leq \ell^2 - \frac{1 - \ell/k}{1 - 1/\ell}, \\ \frac{(\ell - 1)^2}{\ell} &\leq \ell(\ell - 1) + \ell/k - 1, \end{aligned}$$

which clearly holds for all $\ell \geq 2$, $k > 0$. We claim that carrying out Construction 5.18 with $t := \sqrt{(\log_\ell k + 1)/2} - 1$ yields a sequence \mathcal{T} of length at least t . Clearly, this is the case if \mathcal{T} was constructed according to Cases 1 and 2.

Let us show that \mathcal{T} has length t also when it was constructed according to Case 3. To prove this, it suffices to show that we removed at most t duplicate bases. By Lemma 5.19, there is no triangular separation in \mathcal{S} between two sequences \mathcal{S}_1 and \mathcal{S}_2 with the same base. Moreover, \mathcal{S}_1 points left and \mathcal{S}_2 points right. Thus, again by Lemma 5.19, both \mathcal{S}_1 and \mathcal{S}_2 cannot share a base with any other maximal homogeneous subsequence of \mathcal{S} . Hence, the duplicate bases we removed are from pairwise disjoint pairs of maximal homogeneous subsequences. Since there are $2t$ of these sequences, we removed at most t duplicate bases. Hence, \mathcal{T} has length at least t .

Finally, consider the case that \mathcal{T} was constructed according to Case 4. To prove that \mathcal{T} has length at least t , it suffices to show that out of the $\log_\ell k - 1$ separations in \mathcal{S} , there are at most $\log_\ell k - 1 - 2t$ triangular separations. In that case, at least $2t$ separations remain in \mathcal{S} after removing each triangular separation, meaning that there are either at least t separations of order two or at least t separations of order three. Note that each triangular separation is in a hinged subsequence of \mathcal{S} . By Lemma 5.20, each such subsequence is homogeneous or L_1 -trivial. Thus, since Cases 1 and 2 did not apply when constructing \mathcal{T} , each hinged subsequence has length at most t . Furthermore, by Lemma 5.21, there are at most two hinged subsequences that do not consist of homogeneous subsequences and, since Case 3 did not apply, there are at most $2t$ maximal homogeneous subsequences. Thus, overall, there are at most $t(2t + 2)$ triangular separations in \mathcal{S} . Plugging in $t = \sqrt{(\log_\ell k + 1)/2} - 1$ we have

$$\begin{aligned} t(2t + 2) &= \left(\sqrt{(\log_\ell k + 1)/2} - 1 \right) \left(2\sqrt{(\log_\ell k + 1)/2} \right) = \log_\ell k + 1 - 2\sqrt{(\log_\ell k + 1)/2} \\ &= \log_\ell k - 1 - 2t. \end{aligned} \quad \square$$

Verifying Definition 3.1 of well-formed separator sequences. In the remainder of this subsection, we prove that each case of Construction 5.18 indeed yields a well-formed separator sequence, that is, we verify that the properties in Definition 3.1 are satisfied. We consider the cases in order.

Lemma 5.23. If \mathcal{T} was constructed according to Case 1 in Construction 5.18, then \mathcal{T} is a well-formed separator sequence of width three.

Proof. Let (\hat{A}, S, \hat{B}) be in \mathcal{T} , let \mathcal{S}' be a sequence of separations as in Case 1 of Construction 5.18, and let (A, B) be the separation in \mathcal{S}' defining (\hat{A}, S, \hat{B}) . By Condition 1 of Lemma 5.14, \mathcal{S}' is linear and thus, by Corollary 5.17, (C, D) as in Construction 5.18 is the base of each separation in \mathcal{S}' .

To verify Property (i) of well-formed separator sequences, it suffices to observe that, since (A, B) is a separation, one has $A \cup (D \setminus C) \cup (B \setminus (D \setminus C)) = A \cup B = V(G) = B \cup A = B \cup (C \setminus D) \cup (A \setminus (C \setminus D))$. By Construction 5.18, the set $\hat{A} \cup \hat{B}$ equals either the first or the last set in these equations.

To verify Property (ii), for the sake of a contradiction, assume that there is an edge between $\hat{A} \setminus \hat{B}$ and $\hat{B} \setminus \hat{A}$. Consider the case that \mathcal{S}' points left. Then, by the construction of \hat{A} and \hat{B} , there is an edge between $(A \cup (D \setminus C)) \setminus (B \setminus (D \setminus C))$ and $(B \setminus (D \setminus C)) \setminus (A \cup (D \setminus C))$. Note that the first set equals $(A \setminus B) \cup (D \setminus C)$ and the second set equals $(B \setminus A) \setminus (D \setminus C)$. Since (A, B) is a separation, this implies that there is an edge between $D \setminus C$ and $(B \setminus A) \setminus (D \setminus C)$. Since $(D \setminus C) \cup (D \cap C) \cup (C \setminus D) = V(G)$ and $(B \setminus A) \cap (D \cap C) = \emptyset$, this implies that there is an edge between $D \setminus C$ and $C \setminus D$. This is a contradiction to the fact that (C, D) is a separation. The case that \mathcal{S}' points right is analogous.

To verify Property (iii), we have to show that $A \cap B = (A \cup (D \setminus C)) \cap (B \setminus (D \setminus C))$ if \mathcal{S}' points left and that $A \cap B = (B \cup (C \setminus D)) \cap (A \setminus (C \setminus D))$ if \mathcal{S}' points right. However, both cases are trivial since in the first case $A \cap B \cap (D \setminus C) = \emptyset$ and in the second case $A \cap B \cap (C \setminus D) = \emptyset$. Moreover, $|A \cap B| = 3$ since \mathcal{S}' is a triangular sequence.

For Property (iv), assume that there is an element (\hat{A}', S', \hat{B}') of \mathcal{T} succeeding (\hat{A}, S, \hat{B}) and let (A', B') be the separation corresponding to (\hat{A}', S', \hat{B}') . Consider the case that \mathcal{S}' points left. Then $\hat{A} = A \cup (D \setminus C) \subsetneq A' \cup (D \setminus C) = \hat{A}'$ because $A \subsetneq A'$ by Conditions 1 and 3 of Lemma 5.14 and since $A, A' \subseteq C$ because (A, B) and (A', B') point left. Moreover, $A, A' \subseteq C$ we have $B, B' \supseteq D \setminus C$ and hence, $B \cap (D \setminus C) = B' \cap (D \setminus C) = D \setminus C$. Thus, since $B \supsetneq B'$ by Conditions 1 and 3 of Lemma 5.14, $\hat{B} = B \setminus (D \setminus C) \supsetneq B' \setminus (D \setminus C) = \hat{B}'$. The case that \mathcal{S}' points right is analogous.

We claim that \mathcal{T} fulfills Property (vb): For the second part, clearly, S induces a triangle of the required form. To see the first part, assume that (\hat{A}, S, \hat{B}) is the first element of \mathcal{T} . If (A, B) points left, then $B \cap L_1 \subseteq D$ by Lemma 5.11. Since $L_1 \subseteq A \cup B \cap L_1$ we thus have $L_1 \subseteq A \cup (D \setminus C) = \hat{A}$, as required. The case that (A, B) points right is analogous.

Finally, Properties (vi) to (viii) directly follow from the fact that (A, B) is nice and of order three. \square

Lemma 5.24. If \mathcal{T} is constructed according to Case 2 in Construction 5.18, then \mathcal{T} is a well-formed separator sequence of width three.

Proof. Properties (i) to (iii) are fulfilled since \mathcal{S}' is a sequence of triangular separations. The linearity of \mathcal{S}' implies Property (iv).

We claim that \mathcal{T} fulfills Property (vb): Clearly, the intersections $A \cap B$ in the definition of \mathcal{T} induce triangles of the required form. It remains to show $L_1 \subseteq A'$ for the first separation (A', B') in \mathcal{S}' . Assume that this is not the case. Then, since \mathcal{S}' is L_1 -trivial, we have $L_1 \subseteq B'$. Furthermore, since \mathcal{S}' is hinged, the path induced by each separation in \mathcal{S}' touches L_1 in the same place. Thus, $L_1 \subseteq B$ for each separation (A, B) in \mathcal{S}' . Since \mathcal{T} contains $(B, A \cap B, A)$ in this case, it satisfies Property (vb).

Finally, Properties (vi) to (viii) directly follow from the fact that each separation in \mathcal{S}' is nice. \square

For Case 3, we first need to show that all the considered bases are nontrivial and that their induced separators differ.

Lemma 5.25. Each base is nontrivial in the sequence \mathcal{S}' of bases in Case 3 of Construction 5.18 and, moreover, for each pair of bases (C, D) and (C', D') in \mathcal{S}' , we have $C \cap D \neq C' \cap D'$.

Proof. We first prove that, for each pair of bases (C, D) and (C', D') in \mathcal{S}' , we have $C \cap D \neq C' \cap D'$. Observe that (C, D) and (C', D') are the bases of two different maximal homogeneous subsequences \mathcal{T} and \mathcal{T}' of \mathcal{S}' . Towards a contradiction, assume that $C \cap D = C' \cap D'$. Then, \mathcal{T} and \mathcal{T}' point to different directions. Without loss of generality, each separation of \mathcal{T} comes before each separation of \mathcal{T}' in \mathcal{S}' . Let (A, B) and (A', B') be separations in \mathcal{T} and \mathcal{T}' , respectively. Then, (A, B) and (A', B') fulfill the preconditions of Lemma 5.16. This implies that $(C, D) = (C', D')$ by Lemma 5.11 since \mathcal{T} and \mathcal{T}' are homogeneous, and therefore L_1 -nontrivial by definition. We now have our contradiction, since the sequence \mathcal{S}' output by Case 3 does not contain duplicates.

It remains to show that all bases in \mathcal{S}' are nontrivial. Let (A, B) be a separation whose base (C, D) is in \mathcal{S}' . Since (A, B) is part of a homogeneous sequence, it either points left or right and, in particular, is L_1 -nontrivial. Without loss of generality, assume that (A, B) points left, the other case is similar. By Definition 5.12, $A \subseteq C$. Moreover $(A \cap B) \setminus D \neq \emptyset$, whence we have $C \setminus D \neq \emptyset$. By L_1 -nontriviality $(B \setminus A) \cap L_1 \neq \emptyset$. Since, by Lemma 5.11, $B \cap L_1 \subseteq D$, we also have $D \setminus C \neq \emptyset$. Hence, (C, D) is nontrivial. \square

Lemma 5.26. If \mathcal{T} was constructed according to Case 3 in Construction 5.18, then \mathcal{T} is a well-formed separator sequence of width two.

Proof. Clearly, as each base of a separation is itself a separation, Properties (i) and (ii) of well-formed separator sequences are fulfilled. It is easy to see that Property (iii) is fulfilled as well.

To prove Property (iv) first recall that each base in \mathcal{S}' is nontrivial by Lemma 5.25. Let (C, D) be the base in \mathcal{S}' of some separation (A, B) in \mathcal{S} that is not the last one and let (C', D') be the base in \mathcal{S}' belonging to a separation (A', B') with a higher index than (A, B) in \mathcal{S} . We claim that $C \subsetneq C'$ and $D \supsetneq D'$. Since, by definition of nice separations (Definition 5.7), D is uniquely determined once C and $C \cap D$ are defined, whence it suffices to prove that $C \subsetneq C'$. Since both $C \cap D$ and $C' \cap D'$ are edges in G with endpoints in L_1 , they subdivide the region enclosed by L_1 into three regions. One of these regions, R , is incident with $C \cap D$ and not incident with $C' \cap D'$ because $C \cap D \neq C' \cap D'$ by Lemma 5.25. Again, since $C \cap D \neq C' \cap D'$, there is a vertex $v \in (C \cap D) \setminus (C' \cap D')$. Moreover, since (C, D) is nontrivial, v has a neighbor $u \in L_1 \setminus D$ contained in R . We distinguish two cases: vertex u is contained in A or B .

Case 1: $u \in A$. Then, as $A \subsetneq A'$ and $u \in L_1$, we have $u \in C'$ by Lemma 5.11. Furthermore, $u \in C \setminus D$ by the choice of u . Since $C \setminus D$ is connected (Lemma 5.8) and $(C \setminus D) \cap C' \cap D' = \emptyset$, we obtain $C \setminus D \subseteq C'$. Since $C \cap D$ is connected to u via v and $v \notin C \cap D \cap C' \cap D'$, furthermore $C \cap D \subseteq C'$ holds. Hence, if $u \in A$, we have $C \subsetneq C'$.

Case 2: $u \in B$. We lead this case to a contradiction. By Definition 5.7 of nice separations, B is enclosed by the curve induced by the vertices in L_1 that are also in R and a path P contained in $A \cap B$. Since $|(V(P) \cap L_1) \setminus \{v\}| \leq 1$, we have that $|B \cap (C' \cap D')| \leq 1$. This is a contradiction, since $B' \subsetneq B$ and there are two vertices in $C' \cap D' = A' \cap B' \cap L_1$.

This proves Property (iv). Finally, since for each separation (C, D) in \mathcal{T} , we have that $C \cap D$ is an edge and $C \cap D \subseteq L_1$, Properties (vi) to (viii) are fulfilled. \square

Lemma 5.27. If \mathcal{T} was constructed according to Case 4 in Construction 5.18, then \mathcal{T} is a well-formed separator sequence of width two or three.

Proof. Clearly, each object in \mathcal{T} is a separation, hence Properties (i) and (ii) are fulfilled. By the choice of the separations in \mathcal{T} , also Property (iv) holds: for each two consecutive separations (A, B) and (A', B') in \mathcal{S} , the initial sequence in Construction 5.18, we have $A \subsetneq A'$ and $B \supsetneq B'$ by

Condition 3 of Lemma 5.14. Hence, the same holds for the subsequence \mathcal{T} . Property (va) is fulfilled since none of the separations (A, B) in \mathcal{T} induces a triangle, and since $G[A \cap B]$ contains a path whose endpoints are in L_1 by Definition 5.7 of nice separation. Finally, also Properties (vi) to (viii) follow directly from Definition 5.7. \square

Combining Lemmas 5.22 to 5.24, 5.26 and 5.27, we obtain the following corollary. Note that, in the case that G is outerplanar, there are no nice separations of order three in G , and hence, the well-formed separator sequence constructed in Construction 5.18 has width two.

Corollary 5.28. Let G be an r -outerplanar triangulated disk, let $\ell \geq 2$ be an upper bound on the size of each block in $G - L_1$ and $k > 0$ a lower bound on $q(V(G))$. Using Construction 5.18, we can construct a well-formed separator sequence of width at most three and of length at least $\sqrt{(\log_\ell k + 1)/2} - 1$. If G is outerplanar, then the width of the sequences is two instead.

5.3 Proof of Theorem 3.2

Let G be an r -outerplanar triangulated disk with n vertices. If $t := \lfloor \sqrt[2r]{\log(n)/6^r} \rfloor \leq 0$, then the theorem follows trivially. Thus, in the following, assume that $t \geq 1$, that is, $\log(n) \geq 6^{2r^2} \geq 9$. We use induction on the outerplanarity r of G in order to prove that we can construct a well-formed separator sequence of width at most $2r$ and length at least t for G .

For $r = 1$, note that $q(V) = |L_1| = n$ and there are no blocks in $G - L_1$. Hence, by Corollary 5.28, we can construct a well-formed separator sequence of width two and length at least $\sqrt{(\log(n) + 1)/2} - 1$. In this case, $\sqrt{(\log(n) + 1)/2} - 1 \geq t$ is implied by

$$\sqrt{\log(n)/2} - 1 \geq \sqrt{\log(n)}/6, \quad \sqrt{\log(n)} - \sqrt{\log(n)}/3 \geq 2, \quad \text{and} \quad \log(n) \geq 9.$$

Now, assume that the statement is true for $(r - 1)$ -outerplanar triangulated disks, where $r - 1 \geq 1$, and we prove it for r -outerplanar triangulated disks. Assume that there is a block C in $G - L_1$ with at least $s := 2^{(\log(n))^{\frac{r-1}{r}}}$ vertices. It is not hard to see that, since G is a triangulated disk and since C contains at least three vertices, C is a triangulated disk. Moreover, C has outerplanarity at most $r - 1$. Therefore, we can apply the inductive hypothesis to C . We thus infer that there is a well-formed separator sequence \mathcal{S} for C of width at most $2(r - 1)$ and length at least $\lfloor (\log(s))^{1/2(r-1)}/6^{r-1} \rfloor = \lfloor 6 \cdot \sqrt[2r]{\log(n)/6^r} \rfloor \geq 6t$. By Lemma 5.4, we can extend \mathcal{S} to a well-formed separator sequence of width at most $2r$ and length at least t for G .

Now, assume that each block in $G - L_1$ contains at most s vertices. Note that $q(V(G)) \geq |L_1| + (n - |L_1|)/s$, and hence, $q(V) \geq n/s$. By Corollary 5.28, there is a well-formed separator sequence of width at most $3 \leq 2r$ and length at least $\sqrt{(\log_s(q(V)) + 1)/2} - 1$. We claim that this sequence has length at least t . This claim follows from the following list of inequalities that are pairwise equivalent:

$$\begin{aligned} \sqrt{(\log_s(n/s) + 1)/2} - 1 &\geq t, \\ \log_s(n/s) &\geq 2(t + 1)^2 - 1, \\ \frac{\log(n/s)}{\log(s)} &\geq 2(t + 1)^2 - 1, \\ \frac{\log(n)}{\log(s)} &\geq 2(t + 1)^2, \\ \frac{\log(n)}{(\log(n))^{\frac{r-1}{r}}} &\geq 2(t + 1)^2, \\ (\log(n))^{\frac{1}{r}} &\geq 2(t + 1)^2. \end{aligned}$$

Now the last inequality is true because, for $r, t \geq 1$, we have $2(t + 1)^2 \leq 8t^2 \leq \sqrt[2r]{\log(n)}$. \square

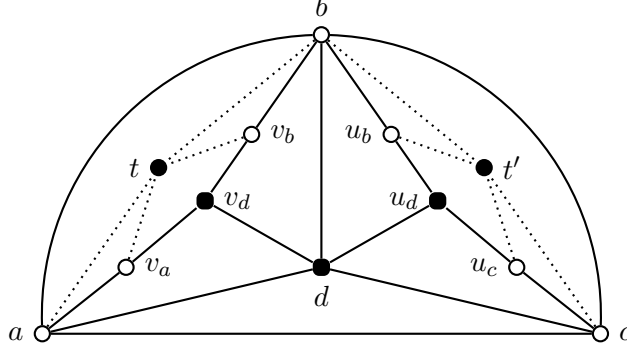


Figure 5: An example showing that twins can be essential for obtaining a (2-outer)planar support. The hyperedges consist of the solid lines in the figure, plus $\{a, v_a, t, t', c\}, \{a, v_b, t, t', c\}, \{b, v_a, t, t', c\}, \{b, v_b, t, t', c\}, \{b, u_b, t, t', a\}, \{b, u_c, t, t', a\}, \{c, u_b, t, t', a\}, \{c, u_c, t, t', a\}$. The vertices t and t' are twins, and \mathcal{H} has a (2-outer)planar support but $\mathcal{H} - t$ does not.

6 Supplement I: Beware of removing twins

As mentioned in Section 1, earlier results about PLANAR SUPPORT (e.g., Mäkinen [20, p. 179], Buchin et al. [5, p. 346], Kaufmann et al. [18, p. 399]) have been obtained under the assumption that the input hypergraph is *twinless*. In Figure 5, we provide a concrete example why removing twins is harmful. The vertex-set of the hypergraph \mathcal{H} shown in Figure 5 is:

$$\mathcal{V} := \{a, b, c, d, v_a, v_b, v_d, u_b, u_c, u_d, t, t'\}.$$

We construct \mathcal{H} in such a way that t and t' are twins and \mathcal{H} has a planar support but $\mathcal{H} - t$ does not. Let \mathcal{E} contain the size-two hyperedges for each *solid* edge shown in Figure 5. Observe that the embedding for this graph, and thus for any support for a hypergraph containing these edges, is basically fixed: The set $\{a, b, c, d\}$ induces a K_4 and any plane embedding of the K_4 has one face for each triangle. Now the path from a to b containing v_d has to be inside the face that is incident with a, b , and c as v_d is a neighbor of d . The same holds for the path from b to c containing u_d . The remaining hyperedges contained in \mathcal{E} are:

$$\begin{array}{llll} \{a, v_a, t, t', c\}, & \{a, v_b, t, t', c\}, & \{b, v_a, t, t', c\}, & \{b, v_b, t, t', c\}, \\ \{b, u_b, t, t', a\}, & \{b, u_c, t, t', a\}, & \{c, u_b, t, t', a\}, & \{c, u_c, t, t', a\}. \end{array}$$

Adding t and t' and the dotted edges to the solid graph gives a planar support for \mathcal{H} .

Now consider the hypergraph $\mathcal{H} - t$. The solid edges are still hyperedges of this hypergraph, hence the embedding of the solid edges and their incident vertices is fixed in any support. Now observe that in any planar support either v_a is not adjacent to b or v_b is not adjacent to a . Moreover, neither of these vertices can be adjacent to c . Thus, to make the graph induced by the hyperedges containing v_a or v_b connected, t' must be adjacent to one of the two vertices in any support. For the same reason, t' must be adjacent to u_b or u_c . This is not possible since each face is either incident with v_a and/or v_b or with u_b and/or u_c but not both. Hence, $\mathcal{H} - t$ has no planar support. Therefore, removing one vertex of a twin class can transform a yes-instance into a no-instance.

The above example can be generalized to make the twin classes arbitrarily large: Copy the vertex set above ℓ times, and let

$$V_i := \{a_i, b_i, c_i, d_i, v_{i,a}, v_{i,b}, v_{i,d}, u_{i,b}, u_{i,c}, u_{i,d}, t_i, t'_i\}$$

denote the vertex set of the i -th copy. Within each copy, add the size-two hyperedges as in the example above. Then, further add a distinguished vertex v^* , and add the size-two

hyperedges $\{a_i, v^*\}$, $\{b_i, v^*\}$, and $\{c_i, v^*\}$ to the hypergraph. After this, for any support G , $G[V_i]$ is constrained to be a copy of Figure 5, and for each copy, v^* is adjacent to the three vertices a_i , b_i , and c_i . Now, let $A := \{a_i \mid 1 \leq i \leq \ell\}$, $B := \{b_i \mid 1 \leq i \leq \ell\}$, and $C := \{c_i \mid 1 \leq i \leq \ell\}$, $V_a := \{v_{i,a} \mid 1 \leq i \leq \ell\}$, $V_b := \{v_{i,b} \mid 1 \leq i \leq \ell\}$, $U_b := \{u_{i,b} \mid 1 \leq i \leq \ell\}$, $U_c := \{u_{i,c} \mid 1 \leq i \leq \ell\}$, and let $T := \{t_i \mid 1 \leq i \leq \ell\} \cup \{t'_i \mid 1 \leq i \leq \ell\}$. Then, add the hyperedges

$$\begin{aligned} A \cup C \cup V_a \cup T \cup \{v^*\}, & \quad A \cup C \cup V_b \cup T \cup \{v^*\}, \\ B \cup C \cup V_a \cup T \cup \{v^*\}, & \quad B \cup C \cup V_b \cup T \cup \{v^*\}, \\ B \cup A \cup U_b \cup T \cup \{v^*\}, & \quad B \cup A \cup U_c \cup T \cup \{v^*\}, \\ C \cup A \cup U_b \cup T \cup \{v^*\}, & \quad C \cup A \cup U_c \cup T \cup \{v^*\}. \end{aligned}$$

The instance is a yes-instance as v^* can be used to “connect” partial solutions for each V_i that are obtained by copying the solution for the simple example. Moreover, each face that is initially incident with $\{a_i, b_i, v_{i,a}, v_{i,b}, v_{i,d}\}$ has to contain at least one vertex of T . Since there are ℓ such faces, removing one vertex of T transforms the yes-instance into a no-instance.

7 Supplement II: Non-uniform fixed-parameter tractability

Theorem 7.1. *Let Π be a graph property that is closed under adding degree-one vertices. There is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that, for each fixed $m \in \mathbb{N}$, there is an algorithm that determines whether a given hypergraph \mathcal{H} with m hyperedges has a support satisfying Π in time $f(m) \cdot \text{poly}(|\mathcal{H}|)$.*

Note that the theorem holds in particular for Π being planarity or r -outerplanarity.

Proof sketch. Let us call a hypergraph Π -supportable if it admits a Π -support. We define a quasi-order \preceq on the family of hypergraphs with m hyperedges such that, if \mathcal{H} is Π -supportable and $\mathcal{H} \preceq \mathcal{G}$, then \mathcal{G} is Π -supportable. We show that, for every $m \in \mathbb{N}$, the family Ψ_m of Π -supportable hypergraphs that are minimal under \preceq is finite.

To define \preceq , we say that $\mathcal{H} \preceq \mathcal{G}$ if \mathcal{H} can be obtained from \mathcal{G} by iteratively removing a vertex that has a twin. If we allow zero removals so that \preceq is reflexive, it is clear that \preceq is a quasi-order. Furthermore, if \mathcal{H} has a Π -support G , then adding the missing twins of a vertex v in \mathcal{G} as degree-one vertices to v in G will yield a Π -support for \mathcal{G} . Thus indeed, if \mathcal{H} is Π -supportable, so is \mathcal{G} .

To see that Ψ_m is finite, consider the representation of an m -hyperedge hypergraph \mathcal{H} as a 2^m -tuple $t_{\mathcal{H}}$, each entry of which represents the size of a distinct twin class. The set of such tuples is quasi-ordered by the natural extension of \leq as $(a_1, \dots, a_{\ell}) \leq (b_1, \dots, b_{\ell})$ if and only if $a_i \leq b_i$ for each $i \in \{1, \dots, \ell\}$. Moreover, Higman [15, Theorem 2.3] has shown that every infinite sequence of 2^m -tuples contains two tuples t_1, t_2 with $t_1 \leq t_2$. Assume that Ψ_m is infinite; then there is an infinite subset Ψ'_m of hypergraphs which have the same (nonempty) twin classes. For hypergraphs \mathcal{H}, \mathcal{G} with the same twin classes, $t_{\mathcal{H}} \leq t_{\mathcal{G}}$ implies $\mathcal{H} \preceq \mathcal{G}$. Thus, Ψ'_m implies an infinite sequence of tuples that are pairwise incomparable under \leq , a contradiction. Hence, Ψ_m is finite.

Finally, to obtain an algorithm for every fixed m as in the theorem, we hard-wire the family Ψ_m of Π -supportable hypergraphs minimal with respect to \preceq into the algorithm. The algorithm simply checks whether its input hypergraph \mathcal{H} fulfills $\mathcal{F} \preceq \mathcal{H}$ for some $\mathcal{F} \in \Psi_m$, which clearly can be checked in polynomial time for each $\mathcal{F} \in \Psi_m$. \square

8 Conclusion

So far, we only used well-formed separator sequences for kernelization. It is interesting to find more algorithmic applications of these separators, for example in a divide and conquer algorithm for PLANAR SUPPORT. We would also like to point out that well-formed separator sequences can be used to find nicely structured separators in r -outerplanar graphs that are *not* triangulated

disks: an r -outerplanar graph G can be turned into a triangulated disk G' such that each vertex remains on its layer [3]. Hence, by computing a long well-formed separator sequence for G' , one obtains for G a separator sequence satisfying Properties (i) to (iv) and (vi) to (viii) of Definition 3.1. Additionally, the graph $G[S_i]$ is a subgraph of an induced path or a cycle. Using this approach, we conjecture that it is also possible to apply our arguments to the variant of PLANAR SUPPORT that asks for a planar support with a minimum number of edges.

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